Cross-Laplacians Based Topological Signal Processing over Cell MultiComplexes

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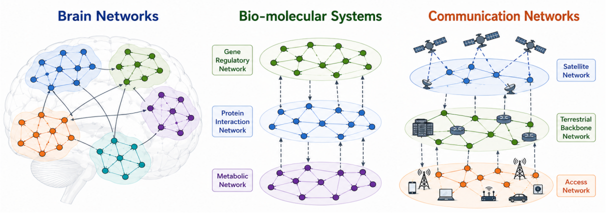
Motivations

Natural and human-made systems involve rich interactions among heterogeneous domains.

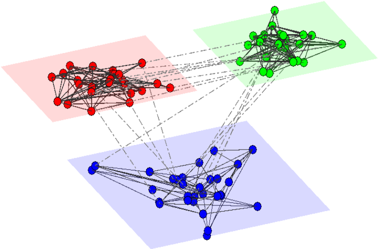


Complex networks: multiple interconnected subsystems that interact through relationships that have different meanings and often operate at different scales.

How can we extract meaningful information from such systems at different levels of resolution?

Multilayer networks

- Multilayer networks: model complex systems through multiple interacting layers of connectivity (S. Boccaletti et al. 2006, M. De Domenico et al. 2013, G. Bianconi 2021...).



Challenge: capturing higher-order interactions between groups of similar or heterogeneous entities in complex networks

- Higher-order multiplex networks: intra-layers interactions in multilayer networks modelled as simplicial complexes (S. Krishnagopal, G. Bianconi 2023)
- E.M. Moutuou et al. (2023) introduced the cross-Laplacian operators as algebraic descriptors of simplicial multi-complex networks.

Contribution

Our novel contributions

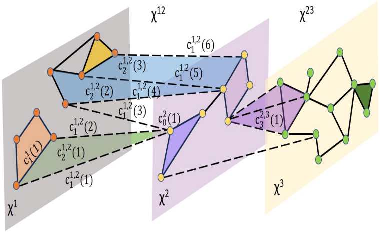
- Introduce Cell MultiComplex (CMC) spaces as novel topological domains for representing higher-order relationships among interconnected complexes.
- Use cross-Laplacian operators as algebraic descriptors of CMCs, enabling the extraction of topological invariants at different resolutions, whether global or local, inter-layer or intra-layer.
- Topological Signal Processing (TSP) over CMC spaces to enable signal processing at different resolution levels.

Cell MultiComplex

Abstract cell complex $\mathcal{C} = \{\mathcal{S}, \prec_b, \dim\}$: a set $\mathcal{S}$ of abstract elements c (cells) provided with a binary relation (incidence) $\prec_b$ and with a dimension function.

Definition

A Cell MultiComplex (CMC) $\mathcal{X}$ is a topological space composed of a finite collection of interdependent cell complexes, each associated with a topological layer. The interdependence among these complexes involves higher-order inter-layer interactions modeled by cross-complexes.



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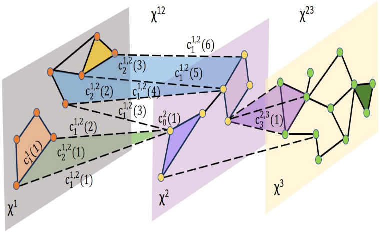


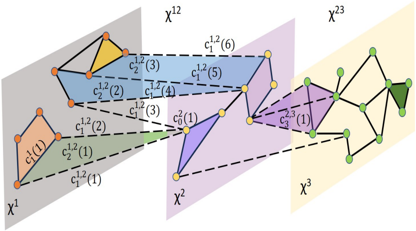
Cross-cells: $c_q^{\ell,m}(i)$, i-th cross-cell of order q between layers ℓ, m.

Cross-cells of order 1 (cross-edges), of order 2 (cross-polygons), of order 3 (cross-polyhedra)...

Cell MultiComplex

Given the cross-cell $c_q^{\ell,m}(i)$, its ℓ-layer faces and m-layer faces are the cells lower bounding $c_q^{\ell,m}(i)$ and belonging to the ℓ- and m-layer, respectively.

The q-order (k,n) cross-cell $c_q^{\ell,m}$ has maximal faces of order k on layer ℓ and faces of order n on layer m.



$c_2^{1,2}(1)$ is a cross-triangle: the face on layer 1 is $c_1^1(1)$, the face on layer 2 is $c_0^2(1)$

From local to global invariants

A CMC can be observed from different perspectives:

- Global perspective: CMC as a monocomplex structure, the Hodge Laplacian for cell complexes is used for topological signal processing (TSP) (S. Sardellitti and S. Barbarossa 2024)
- Novel local view: topological representation of CMCs that can disentangle the local and global homologies to reveal how the topology of one layer affects and controls the topology of others.

The topological properties of a layer depend on the perspective from which it is analyzed:

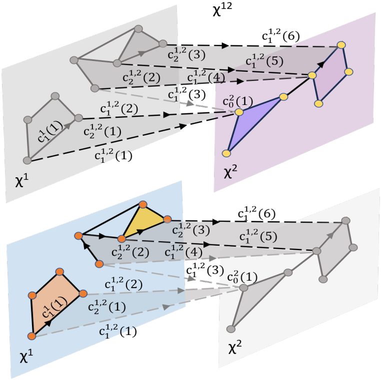
i) from its own viewpoint;
ii) through the lens of other layers;
iii) as part of the overall aggregated structure.

Cross-boundary maps

We introduce the boundary maps of cross-cells in the perspective of a given layer.

Given the cross-complex $\mathcal{X}_{k,n}^{\ell,m}$, we define two distinct cross-boundaries operators for each cross-cell $c_q^{\ell,m} \in \mathcal{X}_{k,n}^{\ell,m}$:

- $\mathbf{B}_{k,n}^{(\ell),m} : C_{k,n} \to C_{k-1,n}$ is a boundary map defined with respect to the crossfaces of order k on layer ℓ
- $\mathbf{B}_{k,n}^{\ell,(m)} : C_{k,n} \to C_{k,n-1}$ is a boundary map with respect to the crossfaces of order n on layer m



Cross-boundary maps

- $\mathbf{B}_{k,n}^{(\ell),m}$: $N_{k-1,n}^{\ell,m} \times N_{k,n}^{\ell,m}$ incidence matrix defined as

$$B_{k,n}^{(\ell),m}(i,j) = \begin{cases} 0, & \text{if } c_{q-1}^{\ell,m}(i) \not\prec_b c_q^{\ell,m}(j) \\ 1, & \text{if } c_{q-1}^{\ell,m}(i) \prec_b c_q^{\ell,m}(j), c_{q-1}^{\ell,m}(i) \sim c_q^{\ell,m}(j) \\ -1, & \text{if } c_{q-1}^{\ell,m}(i) \prec_b c_q^{\ell,m}(j), c_{q-1}^{\ell,m}(i) \nsim c_q^{\ell,m}(j) \end{cases} \tag{1}$$

  where $c_{q-1}^{\ell,m}(i) \in \mathcal{X}_{k-1,n}^{\ell,m}$, $c_q^{\ell,m}(j) \in \mathcal{X}_{k,n}^{\ell,m}$, $\forall i, j$

- $\mathbf{B}_{k,n}^{\ell,(m)}$: $N_{k,n-1}^{\ell,m} \times N_{k,n}^{\ell,m}$ boundary matrix with entries $B_{k,n}^{\ell,(m)}(i,j)$ defined as in (1) and with $c_{q-1}^{\ell,m}(i) \in \mathcal{X}_{k,n-1}^{\ell,m}$, $c_q^{\ell,m}(j) \in \mathcal{X}_{k,n}^{\ell,m}$, $\forall i, j$.

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It holds

$$\mathbf{B}_{k,n}^{(\ell),m}\mathbf{B}_{k+1,n}^{(\ell),m} = \mathbf{0}, \qquad \mathbf{B}_{k,n}^{\ell,(m)}\mathbf{B}_{k,n+1}^{\ell,(m)} = \mathbf{0}.$$

Cross-Laplacians

The (k, n)-cross-Laplacian matrices from layer ℓ:

$$\mathbf{L}_{k,n}^{(\ell),m} = \underbrace{(\mathbf{B}_{k,n}^{(\ell),m})^T \mathbf{B}_{k,n}^{(\ell),m}}_{\text{Lower Laplacian from layer } \ell} + \underbrace{\mathbf{B}_{k+1,n}^{(\ell),m} (\mathbf{B}_{k+1,n}^{(\ell),m})^T}_{\text{Upper Laplacian from layer } \ell}.$$

The (k, n)-cross-Laplacian matrices from layer m:

$$\mathbf{L}_{k,n}^{\ell,(m)} = \underbrace{(\mathbf{B}_{k,n}^{\ell,(m)})^T \mathbf{B}_{k,n}^{\ell,(m)}}_{\text{Lower Laplacian from layer } m} + \underbrace{\mathbf{B}_{k,n+1}^{\ell,(m)} (\mathbf{B}_{k,n+1}^{\ell,(m)})^T}_{\text{Upper Laplacian from layer } m}.$$

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The space $\mathbb{R}^{N_{k,n}}$ admits two different Hodge decompositions:

$$\mathbb{R}^{N_{k,n}} \equiv \operatorname{img}(\mathbf{B}_{k,n}^{(\ell),m\,T}) \oplus \ker(\mathbf{L}_{k,n}^{(\ell),m}) \oplus \operatorname{img}(\mathbf{B}_{k+1,n}^{(\ell),m})$$

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Cross-Betti vectors

The (k,n)-cross-homology groups of $\mathcal{X}$ are:

$$\mathrm{H}_{k,n}^{(\ell)} \cong \ker(\mathbf{L}_{k,n}^{(\ell),m}), \qquad \mathrm{H}_{k,n}^{(m)} \cong \ker(\mathbf{L}_{k,n}^{\ell,(m)}).$$

The (k,n)-cross-Betti vector of $\mathcal{X}_{k,n}^{\ell,m}$ is defined as:

$$\boldsymbol{\beta}_{k,n}^{\ell,m} = [\beta_{k,n}^{(\ell)}, \beta_{k,n}^{(m)}]$$

where the (k,n)-cross-Betti numbers are:

$$\beta_{k,n}^{(\ell)} = \dim(\ker(\mathbf{L}_{k,n}^{(\ell),m})), \qquad \beta_{k,n}^{(m)} = \dim(\ker(\mathbf{L}_{k,n}^{\ell,(m)})).$$

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The cross-Betti numbers capture the homologies of the CMCs.

Construct the cross-Laplacian to capture the desired topological invariant.

Cross-Laplacians in the cross-edges space

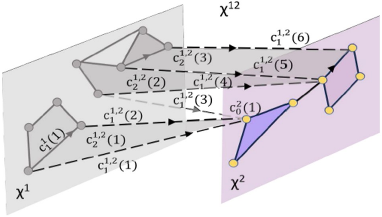
The cross-Laplacians $\mathbf{L}_{0,0}^{(\ell),m}$ is a matrix indexed on the cross-edges $c_1^{\ell,m} \in \mathcal{X}_{0,0}^{\ell,m}$:

$$\mathbf{L}_{0,0}^{(\ell),m} = \underbrace{(\mathbf{B}_{0,0}^{(\ell),m})^T \mathbf{B}_{0,0}^{(\ell),m}}_{\text{Lower Laplacian from layer } \ell} + \underbrace{\mathbf{B}_{1,0}^{(\ell),m} (\mathbf{B}_{1,0}^{(\ell),m})^T}_{\text{Upper Laplacian from layer } \ell}.$$

- The entry (i, j) of the lower Laplacian is equal to ±1 if the cross-edges $c_1^{\ell,m}(i)$ and $c_1^{\ell,m}(j)$ share a vertex on layer m.
- The upper Laplacian identifies the upper adjacencies of the cross-edges $c_1^{\ell,m}$ as boundaries of $(1, 0)$ 2-order cells.

The $(0, 0)$ cross-Laplacian from layer m:

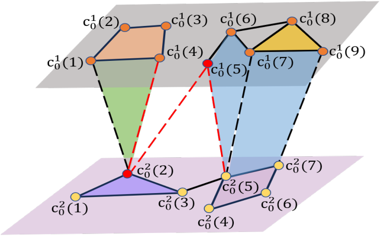
$$\mathbf{L}_{0,0}^{\ell,(m)} = (\mathbf{B}_{0,0}^{\ell,(m)})^T \mathbf{B}_{0,0}^{\ell,(m)} + \mathbf{B}_{0,1}^{\ell,(m)} (\mathbf{B}_{0,1}^{\ell,(m)})^T$$



The Cross-Betti vector $\boldsymbol{\beta}_{0,0}$

Cones: the shortest paths of length 2 between nodes within one layer, passing through a node on the other layer and not belonging to the cross-boundary of 2-order cross-cells.

Closed cone: forms a cycle with intra-layer edges
Open cone: it has a vertex on one layer connecting two unconnected clusters on the other layer



The $(0,0)$ cross-Betti vector $\boldsymbol{\beta}_{0,0}^{\ell,m} = (\beta_{0,0}^{(\ell)}, \beta_{0,0}^{(m)})$ counts the cones open and closed between layers ℓ and m:

- $\beta_{0,0}^{(\ell)} = \dim(\ker(\mathbf{L}_{0,0}^{(\ell),m}))$ counts the cones with vertices on layer m
- $\beta_{0,0}^{(m)} = \dim(\ker(\mathbf{L}_{0,0}^{\ell,(m)}))$ counts those with vertices on layer ℓ.

These vertices are called harmonic cross-hubs.

Signal Processing over CMCs

Given a 2-order CMC $\mathcal{X} = (\mathcal{V}, \mathcal{E}, \mathcal{C})$, we define signals over the set of nodes, edges and 2-cells as

$$\boldsymbol{s}_0 : \mathcal{V} \to \mathbb{R}^N, \boldsymbol{s}_1 : \mathcal{E} \to \mathbb{R}^E, \boldsymbol{s}_2 : \mathcal{C} \to \mathbb{R}^C.$$

The signals $\boldsymbol{s}_1^{\ell,m} \subseteq \boldsymbol{s}_1$ and $\boldsymbol{s}_2^{\ell,m} \subseteq \boldsymbol{s}_2$ are the entries of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ associated with cross-edges and 2-order cross-cells between layer ℓ, m.

Let us focus on the $(0,0)$-cross Laplacian $\mathbf{L}_{0,0}^{(\ell),m} = \mathbf{U}_{0,0}^{(\ell),m} \boldsymbol{\Lambda}_{0,0}^{(\ell),m} \mathbf{U}_{0,0}^{(\ell),m\,T}$.

CMC Fourier Transform of $\boldsymbol{s}_1^{\ell,m}$: $\hat{\boldsymbol{s}}_1^{\ell,m} := \mathbf{U}_{0,0}^{(\ell),m\,T} \boldsymbol{s}_1^{\ell,m}$

Inverse CMC Fourier Transform: $\boldsymbol{s}_1^{\ell,m} := \mathbf{U}_{0,0}^{(\ell),m} \hat{\boldsymbol{s}}_1^{\ell,m}$

Local representation: $\boldsymbol{s}_1^{\ell,m}$ is a vector belonging to $\mathbb{R}^{N_{0,0}^{\ell,m}}$ with $\mathbb{R}^{N_{0,0}^{\ell,m}} \subset \mathbb{R}^E$!

Signal Processing over CMCs

The cross-edge signal $\boldsymbol{s}_1^{\ell,m} \in \mathbb{R}^{N_{0,0}^{\ell,m}}$ can be decomposed as

$$\boldsymbol{s}_1^{\ell,m} = \mathbf{B}_{0,0}^{(\ell),m\,T} \boldsymbol{s}_0^m + \mathbf{B}_{1,0}^{(\ell),m} \boldsymbol{s}_2^{\ell,m} + \boldsymbol{s}_{1,H}^{\ell,m}$$

- $\mathbf{B}_{0,0}^{(\ell),m\,T} \boldsymbol{s}_0^m$: cross-edge flow with zero-circulation along the cross-edges of 2-order $(1,0)$ cross-cells
- $\mathbf{B}_{1,0}^{(\ell),m} \boldsymbol{s}_2^{\ell,m}$ has zero-sum on the vertices over layer m
- $\boldsymbol{s}_{1,H}^{\ell,m} \in \ker(\mathbf{L}_{0,0}^{(\ell),m})$ is the harmonic cross-edge signal.

We define:

- $\mathrm{div}_{cr}(\boldsymbol{s}_1^{\ell,m}) = \mathbf{B}_{0,0}^{(\ell),m} \boldsymbol{s}_1^{\ell,m}$: node signal measuring the conservation of the cross-flows over the nodes of layer m
- $\mathrm{curl}_{cr}(\boldsymbol{s}_1^{\ell,m}) = \mathbf{B}_{1,0}^{(\ell),m\,T} \boldsymbol{s}_1^{\ell,m}$ is a measure of the flow conservation along cross-edges bounding 2-order $(1,0)$ cross-cells.

Filtering signals over CMCs

Goal: estimate cross-edge signals from the noisy observations $\boldsymbol{y}_1^{\ell,m} = \boldsymbol{s}_1^{\ell,m} + \boldsymbol{n}_1$, where $\boldsymbol{n}_1$ is additive noise.

Optimization problem:

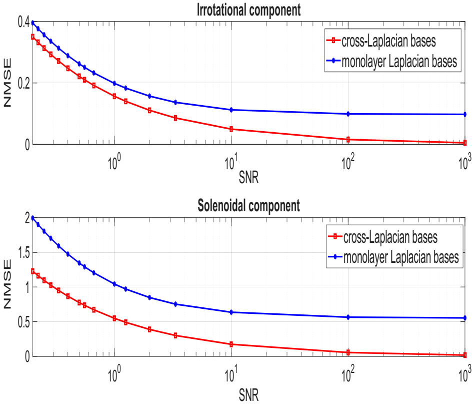
$$\min_{\substack{\boldsymbol{s}_0^{\ell} \in \mathbb{R}^{N_\ell}, \boldsymbol{s}_2^{\ell,m} \in \mathbb{R}^{N_{0,1}^{\ell,m}} \\ \boldsymbol{s}_{1,H}^{\ell,m} \in \mathbb{R}^{N_{0,0}^{\ell,m}}}} \quad \| \mathbf{B}_{0,0}^{\ell,(m)\,T} \boldsymbol{s}_0^{\ell} + \mathbf{B}_{0,1}^{\ell,(m)} \boldsymbol{s}_2^{\ell,m} + \boldsymbol{s}_{1,H}^{\ell,m} - \boldsymbol{y}_1^{\ell,m} \|^2$$

$$\text{s.t.} \qquad \mathbf{B}_{0,0}^{\ell,(m)} \boldsymbol{s}_{1,H}^{\ell,m} = \mathbf{0}, \quad \mathbf{B}_{0,1}^{\ell,(m)\,T} \boldsymbol{s}_{1,H}^{\ell,m} = \mathbf{0}$$

Closed-form solutions:

$$\begin{aligned}
\bar{\boldsymbol{s}}_0^{\ell} &= (\mathbf{B}_{0,0}^{\ell,(m)} \mathbf{B}_{0,0}^{\ell,(m)\,T})^{\dagger} \mathbf{B}_{0,0}^{\ell,(m)} \boldsymbol{y}_1^{\ell,m} \\
\bar{\boldsymbol{s}}_2^{\ell,m} &= (\mathbf{B}_{0,1}^{\ell,(m)\,T} \mathbf{B}_{0,1}^{\ell,(m)})^{\dagger} \mathbf{B}_{0,1}^{\ell,(m)\,T} \boldsymbol{y}_1^{\ell,m} \\
\bar{\boldsymbol{s}}_{1,H}^{\ell,m} &= \boldsymbol{y}_1^{\ell,m} - \mathbf{B}_{0,0}^{\ell,(m)\,T} \bar{\boldsymbol{s}}_0^{\ell} - \mathbf{B}_{0,1}^{\ell,(m)} \bar{\boldsymbol{s}}_2^{\ell,m}
\end{aligned}$$

Numerical results



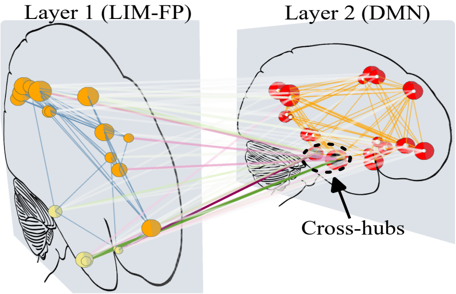
Normalized mean squared error vs SNR

The cross-Laplacian based representation ensures better performance in terms of recovering error for both the cross-irrotational and cross-solenoidal signals.

Application: Brain Networks

Application: Brain network analysis

Goal: investigate the presence of trimers, i.e., pairs of (inter-modules) links incident on a common root region (meta-hub) controlling the inter-modules relations (L. M. Arbabyazd, D. Lombardo, et al. 2020).



First module: $N_1 = 18$ nodes from the Limbic (LIM) subnetwork (khaki) and the Frontoparietal (FP) subnetwork (orange).

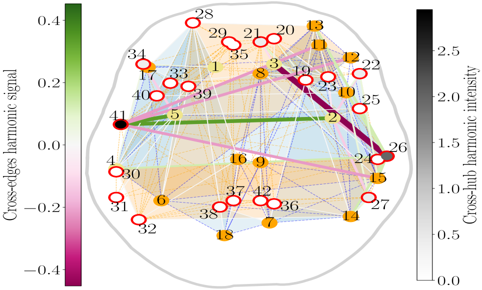
Second module: $N_2 = 24$ nodes from the Default Mode Network (DMN) (red nodes).

Application: Brain network

Proposed two-step method:

1) Topology Learning

$$\begin{aligned} \min_{\mathbf{a}\in\{0,1\}^{N_{1,0}^{\ell,m}}} \quad & \sum_{k=1}^{N_{1,0}^{\ell,m}} a_k \mathrm{tr}\{(\mathbf{X}_{1,0}^{\ell,m})^T \mathbf{b}_{1,0}^{(\ell),m}(k)(\mathbf{b}_{1,0}^{(\ell),m}(k))^T \mathbf{X}_{1,0}^{\ell,m}\} \\ \text{s.t.} \quad & \| \mathbf{a} \|_0 = N_c \end{aligned}$$

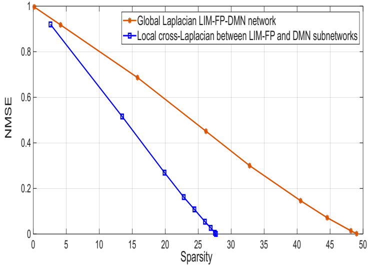


Cross-edge harmonic signal and cross-hub harmonic intensity.

Numerical results

2) Find sparse signal representation with data-fitting error constraint:

$$\begin{array}{ll} \min\limits_{\hat{\boldsymbol{s}}_1^{\ell,m} \in \mathbb{R}^{N_{0,0}^{\ell,m}}} & \parallel \hat{\boldsymbol{s}}_1^{\ell,m} \parallel_1 \\ \text{s.t.} & \parallel \boldsymbol{y}_1^{\ell,m} - \mathbf{U}_{0,0}^{(\ell),m} \hat{\boldsymbol{s}}_1^{\ell,m} \parallel_F \leq \epsilon_{0,0} \end{array}$$



NMSE versus signal sparsity, considering as cross-edge signal dictionaries the eigenvectors of the monolayer Laplacian (LIM-FP-DMN) and of the cross-Laplacian $\mathbf{L}_{0,0}^{(\ell),m}$.

Conclusions

- Cell MultiComplexes (CMCs): novel topological spaces capable of capturing both intra- and inter-layers higher-order interactions across different networks.

- It has been shown how cross-Laplacian matrices are effective algebraic descriptors for representing signals over CMCs.

- We developed a signal processing framework on CMCs to represent and analyse local signals.

- Future developments: extend the signal processing framework to the (k, n)-cross Laplacians to capture novel topological invariants.