Nonstationary Graph Filters Based on Localized Frames

Graph Signal Processing Workshop 2026

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Introduction

Motivation:

- Graph filters fundamental for dealing with data on networks
- Most filter designs restricted by stationarity or complexity

Contributions:

- Localized graph frame (LGF) filter: nonstationary, fast, intuitive
- Implementations, computational complexity
- Filter properties
- Exemplary application: denoising

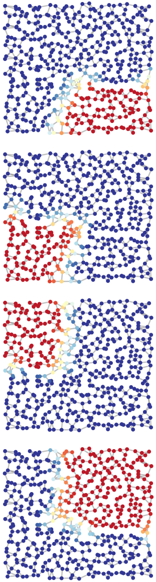
LGF Definition

- Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, \ldots, N\}$ and edge set $\mathcal{E}$
- Weighted adjacency matrix $\mathbf{W} \in \mathbb{R}^{N \times N}$
- Window functions $\mathbf{g}_m$, $m = 1, \ldots, M$, satisfying $\sum_{m=1}^{M} g_m^2[n] \equiv 1$
- Support $\mathcal{S}_m = \{n \colon\, g_m[n] \neq 0\}$ of size $N_m = |\mathcal{S}_m|$ induces subgraph $\mathcal{G}_m$
- Laplacian eigenvectors $\mathbf{u}_{m1}, \ldots, \mathbf{u}_{mN_m}$ of $\mathcal{G}_m$
- Windowed graph Fourier vectors

$$\mathbf{f}_{ml} = \mathbf{g}_m \odot \mathbf{S}_m \mathbf{u}_{ml}, \; m = 1, \ldots, M, \quad l = 1, \ldots, N_m,$$

  with zero-padding matrix $\mathbf{S}_m \in \{0, 1\}^{N \times N_m}$
- Normalized tight frame (i.e., LGF) defined by

$$\mathbf{F} = (\mathbf{f}_{11}, \ldots, \mathbf{f}_{1N_1}, \ldots, \mathbf{f}_{M1}, \ldots, \mathbf{f}_{MN_m}) \in \mathbb{R}^{N \times N'}, \quad N' = \sum_{m=1}^{M} N_m \geq N$$



LGF Properties

- (Over-)complete local Fourier basis
- Strictly localized in vertex domain: low complexity, distributed processing
- Well localized in (global) GFT domain
- N-dimensional subspace $\mathcal{F} = \text{span}\{\mathbf{F}^{\mathsf{T}}\} \subseteq \mathbb{R}^{N'}$ of admissible coefficient sequences $\tilde{x}_{ml} = \mathbf{f}_{ml}^{\mathsf{T}}\mathbf{x}$
- Frame Gramian $\mathbf{P}_f = \mathbf{F}^{\mathsf{T}}\mathbf{F} \in \mathbb{R}^{N' \times N'}$ equals rank-N orthogonal projection matrix onto $\mathcal{F}$

LGF Filter

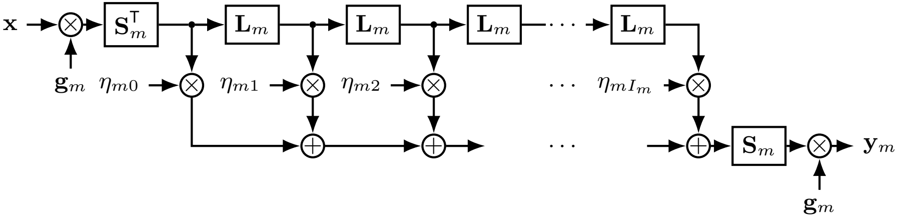
- Weights: $\mathbf{h} = (h_{11} \dots h_{1N_1} \dots h_{M1} \dots h_{MN_M})^\mathsf{T}$ and $\mathbf{D}_h = \text{Diag}(\mathbf{h})$
- Filter stages:

1) Analysis:	$\tilde{\mathbf{x}} = \mathbf{F}^\mathsf{T}\mathbf{x},$	$\tilde{x}_{ml} = \mathbf{f}_{ml}^\mathsf{T}\mathbf{x}$
2) Weighting:	$\tilde{\mathbf{y}} = \mathbf{D}_h\tilde{\mathbf{x}},$	$\tilde{y}_{ml} = h_{ml}\tilde{x}_{ml}$
3) Synthesis:	$\mathbf{y} = \mathbf{F}\tilde{\mathbf{y}},$	$\mathbf{y} = \sum_{m=1}^{M}\sum_{l=1}^{N_m}\tilde{y}_{ml}\mathbf{f}_{ml}$

- Linear symmetric graph filter $\mathbf{y} = \mathbf{H}\mathbf{x}$ with $\mathbf{H} = \mathbf{F}\mathbf{D}_h\mathbf{F}^\mathsf{T} = \mathbf{H}^\mathsf{T}$
- In general, $\tilde{\mathbf{y}} \notin \mathcal{F}$, but $\mathbf{F}\tilde{\mathbf{y}} = \underset{\mathbf{y}}{\operatorname{argmin}} \left\|\mathbf{F}^\mathsf{T}\mathbf{y} - \tilde{\mathbf{y}}\right\|_2^2$
- Nonstationary: $\mathbf{H}$ does not commute with (global) graph Laplacian

Complexity

- Analysis-Weighting-Synthesis: $\mathcal{O}\left(\sum_{m=1}^{M} N_m^2\right)$
- If only $J_m \leq N_m$ weights h_{ml} are nonzero (e.g., local bandpass): $\mathcal{O}\left(\sum_{m=1}^{M} J_m N_m\right)$
- Tapped Laplacian line of order I_m, average local degree $\bar{d}_m$: $\mathcal{O}\left(\sum_{m=1}^{M} N_m \bar{d}_m I_m\right)$



Special Cases

- Identity: $h_{ml} \equiv 1 \implies \mathbf{H} = \mathbf{I}$.
- GFT multipliers (stationary): $M = 1$, $\mathbf{g}_1 = \mathbf{1}$
- Vertex-domain multiplier: $M = N$ and $\mathbf{g}_m = \mathbf{e}_m \implies \mathbf{F} = \mathbf{I}$ and $\mathbf{H} = \mathbf{D}_h$
- Constant subgraph gain ($h_{ml} = c_m$): $\mathbf{H} = \sum_{m=1}^{M} c_m \text{Diag}^2(\mathbf{g}_m)$
- Projection: $h_{ml} \in \{0, 1\} \implies \mathbf{H}^2 \approx \mathbf{H}$

$(\epsilon, \mathbf{F})$-Compatibility

- Can we draw conclusions about $\mathbf{H}$ from weights $\mathbf{h}$?
- Yes, if filtered LGF sequence $\tilde{\mathbf{y}} = \mathbf{D}_h \tilde{\mathbf{x}}$ approximately remains within $\mathcal{F}$

Definition: A weight function $\mathbf{h}$ is $(\epsilon, \mathbf{F})$-compatible if $\|\mathbf{P}_f \mathbf{D}_h \mathbf{P}_f - \mathbf{D}_h\|_2 \leq \epsilon \, \|\mathbf{D}_h\|_2$

- Energy outside $\mathcal{F}$: $\|(\mathbf{I} - \mathbf{P}_f)\tilde{\mathbf{y}}\|_2 \leq \epsilon \, \|\mathbf{D}_h\|_2 \|\mathbf{x}\|_2 \quad \Longrightarrow \quad \mathbf{P}_f^{\perp} \tilde{\mathbf{y}} \approx 0$ for small ϵ
- Intuition: h_{ml} varies slowly between frequencies in same & overlapping subgraphs

Desirable Properties I

Let $\lambda_{\min}$ and $\lambda_{\max}$ be the smallest and largest eigenvalue of $\mathbf{H}$, respectively

Define $h_{\min} = \min\{h_{ml}\}$ and $h_{\max} = \max\{h_{ml}\}$

- Eigenvalue bounds: $h_{\min} \leq \lambda_{\min} \leq \lambda_{\max} \leq h_{\max}$, equality iff $\epsilon = 0$
- Definiteness: $h_{\min} \geq 0 \quad \Longrightarrow \quad \mathbf{H}$ positive semi-definite
- Spectral norm: $\|\mathbf{H}\|_2 \leq \max\{|h_{\max}|, |h_{\min}|\}$
- Frobenius norm: $(1-\epsilon)\|\mathbf{h}\|_2^2 \leq \|\mathbf{H}\|_\mathsf{F}^2 \leq (1+\epsilon)\|\mathbf{h}\|_2^2$

Desirable Properties II

Define $\mathbf{H}_1 = \mathbf{F}\mathbf{D}_{h_1}\mathbf{F}^\mathsf{T}$, $\mathbf{H}_2 = \mathbf{F}\mathbf{D}_{h_2}\mathbf{F}^\mathsf{T}$, and $\mathbf{H}_\mathsf{s} = \mathbf{F}\mathbf{D}_{h_2}\mathbf{D}_{h_1}\mathbf{F}^\mathsf{T}$

- Inner product: $\left|\mathrm{tr}\{\mathbf{H}_1^\mathsf{T}\mathbf{H}_2\} - \mathbf{h}_1^\mathsf{T}\mathbf{h}_2\right| \leq \epsilon\,\|\mathbf{h}_1\|_2\|\mathbf{h}_2\|_2$
- Serial connection: $\|\mathbf{H}_2\mathbf{H}_1 - \mathbf{H}_\mathsf{s}\|_2 \leq \epsilon\,\|h_1\|_\infty\|h_2\|_\infty$
- Commutativity: $\|\mathbf{H}_2\mathbf{H}_1 - \mathbf{H}_1\mathbf{H}_2\|_2 \leq 2\epsilon\,\|h_1\|_\infty\|h_2\|_\infty$

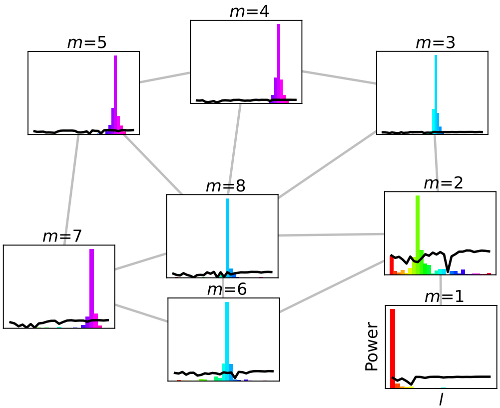
For $h_{ml} \neq 0$ define $\mathbf{H}_{\mathrm{inv}} = \mathbf{F}\mathbf{D}_h^{-1}\mathbf{F}^\mathsf{T}$

- Inversion: $\|\mathbf{H}\mathbf{H}_{\mathrm{inv}} - \mathbf{I}\|_2 \leq \epsilon\max\{|h_{ml}|\}/\min\{|h_{ml}|\}$

Conclusion: ϵ-compatible weights h_{ml} approximately determine LGF filter properties and operations

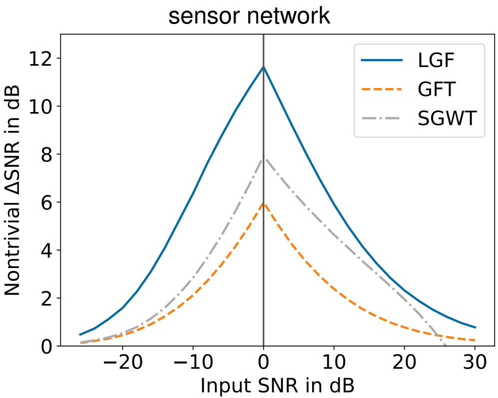
Numerical Experiments: Denoising

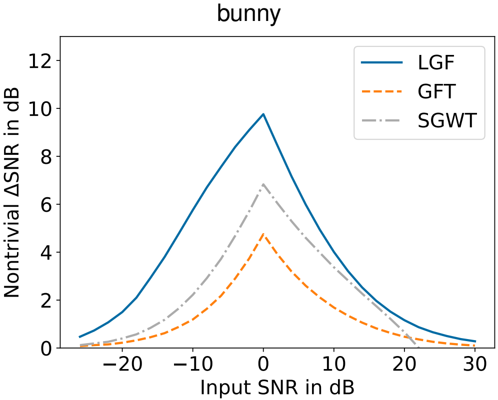
- Estimate signal $\mathbf{s}$ from noisy observation $\mathbf{x} = \mathbf{s} + \mathbf{e}$
- Graph signal $\mathbf{s}$ is nonstationary with varying local bandwidth and center frequency
- Graphs: kNN construction with $k = 3$ for
  ...sensor network (uniformly distributed sensors)
  ...Stanford bunny ($N = 2503$)
- LGF Wiener filter: $h_{ml} = \dfrac{\mathsf{E}\{(\mathbf{f}_{ml}^{\mathsf{T}}\mathbf{s})^2\}}{\mathsf{E}\{(\mathbf{f}_{ml}^{\mathsf{T}}\mathbf{s})^2\} + \sigma^2\|\mathbf{f}_{ml}\|^2}$
- Comparison with Wiener filters in
  ...(global) GFT domain
  ...SGWT domain (Meyer, 3 scales)



hypergraph for overlapping subgraphs

Impact of SNR

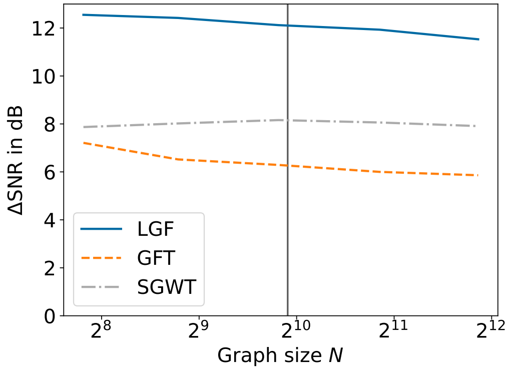


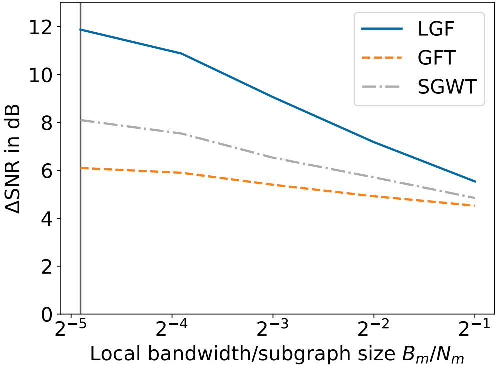


- $N = 961$, $M \approx 25$, local bandwidth $\lfloor N_m/30 \rfloor$
- LGF filter outperforms other filter designs
- LGF speedup: $\times 9$ SGWT, $\times 20$ GFT

- $N = 2503$, same partition size & bandwidth
- Similar performance
- LGF speedup: $9\times$ SGWT, $47\times$ GFT

Impact of Graph Size and Bandwidth (Sensor Network)

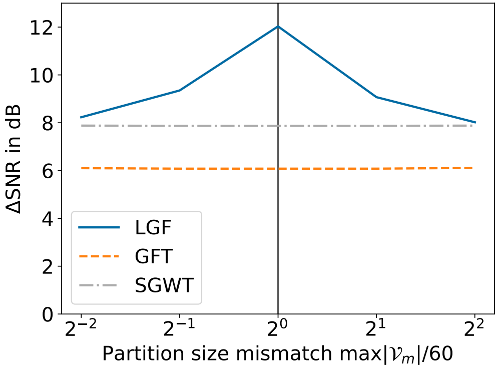


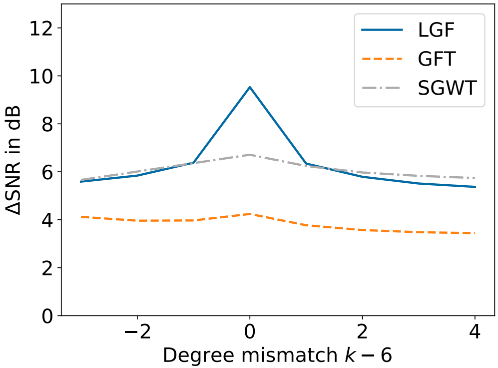


- Partition size $N/M \approx 40$, 0 dB input SNR
- Performance almost constant
- LGF speedup: up to $9\times$ SGWT and $67\times$ GFT

- $N = 961$, $M \approx 25$, 0 dB input SNR
- LGF advantage vanishes at high bandwidth
  - stronger overlap of signal & noise
  - signal more stationary

Robustness





- Topology mismatch between signal model & filters
- LGF performance advantage vanishes
- LGF speedup remains substantial

Conclusion

- LGF filters
  - inspired by Gabor multipliers
  - builds on localized graph frames
  - LGF analysis/weighting/synthesis
  - inherently nonstationary
  - flexible design
  - fast (and distributed) implementations
- $(\epsilon, \mathbf{F})$-compatibility ensures that filter inherits properties from weight function
- Denoising performance superior to GFT and SGWT

Thank you for your attention!

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