Graphs in Financial Markets

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Keynote Talk

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Outline

Graph Research Collaborators

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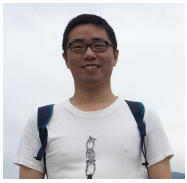
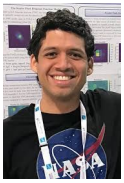


Jiaxi Ying



Vinícius Cardoso

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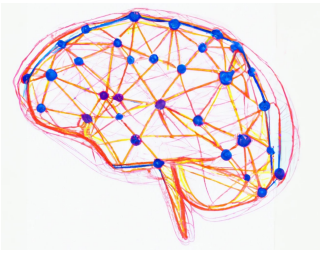
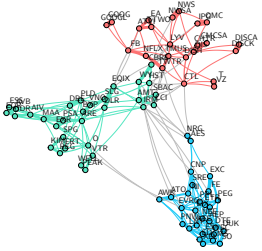
Outline

Graphs Everywhere

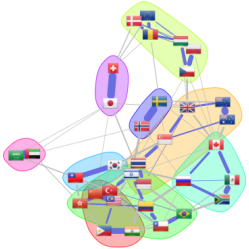
Social network



Brain connectivity



Financial stocks

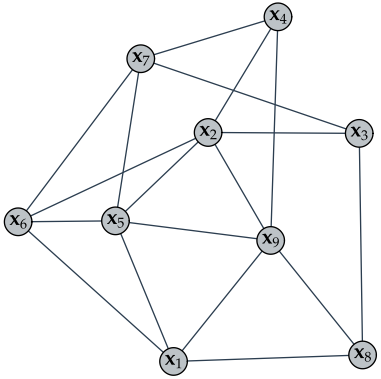


Foreign exchange

Graphical Models

Graphical models are a way to represent and visualize the **relationships** between entities:

- **nodes** correspond to the entities (variables)
- **edges** encode the relationships between entities (dependencies between the variables)
- **edge weights** $\boldsymbol{W}$ encode the strength of the relationships



Graph Matrices

The **adjacency matrix** $\boldsymbol{W}$ directly characterizes a graph: W_{ij} contains the weight of the edge between nodes i and j.

The **degree matrix** $\boldsymbol{D}$ is a diagonal matrix containing the node degrees (row sums of $\boldsymbol{W}$):

$$\boldsymbol{D} = \mathrm{Diag}(\boldsymbol{W}\mathbf{1}).$$

The **Laplacian** $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{W}$ is the key matrix in graph signal processing:

- *Physical interpretation*: measures the **smoothness** of a graph signal $\boldsymbol{x} = (x_1, \ldots, x_p)$:

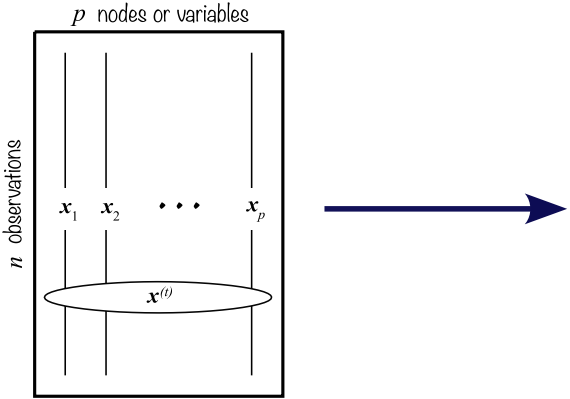
$$\boldsymbol{x}^\top \boldsymbol{L}\, \boldsymbol{x} = \frac{1}{2} \sum_{i,j} W_{ij}(x_i - x_j)^2$$

- *Mathematical properties*: symmetric and positive semidefinite ($\boldsymbol{L} \succeq \mathbf{0}$); singular with $\boldsymbol{L}\mathbf{1} = \mathbf{0}$

Outline

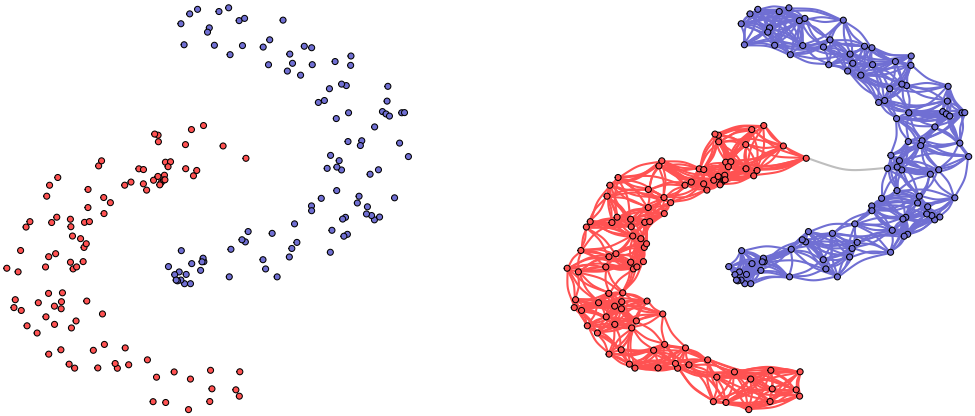
Learning Graphs from Data

- Given a data matrix $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_p] \in \mathbb{R}^{T \times p}$, each column $\boldsymbol{x}_i \in \mathbb{R}^T$ contains T observations from node i (and each row is one observation of the **graph signal**).
- The goal is to obtain a **graph representation** of the data.



Graph Learning Illustration

Example of learning a graph from a dataset with points in $\mathbb{R}^2$:



Overview references: (Mateos et al. 2019) and (Dong et al. 2019).

Learning Graphs from Smooth Signals

- Since $\boldsymbol{x}^\mathsf{T}\boldsymbol{L}\boldsymbol{x}$ measures the smoothness of a graph signal $\boldsymbol{x}$, the total smoothness over T observations collected along the rows of matrix $\boldsymbol{X}$ is $\mathrm{Tr}(\boldsymbol{X}\boldsymbol{L}\boldsymbol{X}^\mathsf{T})$.
- Learn the graph by finding the Laplacian that makes the signals smooth (Kalofolias 2016):

$$\begin{aligned} \underset{\boldsymbol{L}\succeq\boldsymbol{0}}{\text{minimize}} \quad & \mathrm{Tr}\Big(\boldsymbol{X}\boldsymbol{L}\boldsymbol{X}^\mathsf{T}\Big) - \alpha\,\boldsymbol{1}^\mathsf{T}\log(\mathrm{diag}(\boldsymbol{L})) + \tfrac{\beta}{2}\|\boldsymbol{L}\|_{F,\text{off}}^2 \\ \text{subject to} \quad & \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0}, \quad L_{ij} = L_{ji} \leq 0, \quad \forall\, i \neq j. \end{aligned}$$

  where $\mathrm{diag}(\boldsymbol{L})$ contains the node degrees, so $-\log(\mathrm{diag}(\boldsymbol{L}))$ penalizes zero degrees (no isolated nodes); and $\|\boldsymbol{L}\|_{F,\text{off}}^2$ penalizes nodes with very few connections.
- The same formulation can be equivalently rewritten in terms of the adjacency matrix $\boldsymbol{W}$ instead of $\boldsymbol{L}$.
- Other formulations: (Dong et al. 2015) and (Nie et al. 2016).

Graphical Lasso

- **MLE of the covariance matrix Σ** from T observations (with sample covariance $\boldsymbol{S}$):

$$\underset{\boldsymbol{\Sigma} \succ \mathbf{0}}{\text{maximize}} \quad \log \det(\boldsymbol{\Sigma}^{-1}) - \text{Tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{S}).$$

- The **precision matrix** $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}$ encodes partial correlations: $\Theta_{ij} = 0$ implies nodes i and j are conditionally independent given all others — this defines the graph edge structure.

- **Graphical LASSO (GLASSO)** adds an ℓ_1 penalty to promote sparsity in $\boldsymbol{\Theta}$ (Meinshausen and Bühlmann 2006):

$$\underset{\boldsymbol{\Theta} \succ \mathbf{0}}{\text{maximize}} \quad \log \det(\boldsymbol{\Theta}) - \text{Tr}(\boldsymbol{\Theta}\boldsymbol{S}) - \rho\|\boldsymbol{\Theta}\|_{1,\text{off}},$$

where ρ controls the sparsity level.

Learning Graphs from a Gaussian Markov Random Field

- Modeling graph signals as a **Gaussian Markov Random Field (GMRF)**, the MLE of the graph Laplacian is (Lake and Tenenbaum 2010; Egilmez, Pavez, and Ortega 2017; Zhao et al. 2019):

$$\begin{array}{ll} \underset{\boldsymbol{L}\succeq\mathbf{0}}{\text{maximize}} & \log \text{gdet}(\boldsymbol{L}) - \text{Tr}(\boldsymbol{S}\boldsymbol{L}) - \rho\|\boldsymbol{L}\|_{1,\text{off}} \\ \text{subject to} & \boldsymbol{L}\mathbf{1} = \mathbf{0}, \quad L_{ij} = L_{ji} \leq 0, \quad \forall\, i \neq j, \end{array}$$

  where $\boldsymbol{S}$ is the sample covariance matrix.
- Surprisingly, the ℓ_1-norm regularization $\|\boldsymbol{L}\|_{1,\text{off}}$ does **not** promote sparsity — a counter-intuitive result (Ying, M. Cardoso, and Palomar 2020, 2021).
- Instead, one must use a **concave regularizer** to achieve sparse graphs.
- A concave regularizer leads to a nonconvex problem, but it can be solved iteratively via **majorization-minimization (MM)**, yielding a **reweighted ℓ_1-norm** method (Kumar et al. 2020; Ying, M. Cardoso, and Palomar 2020).

Trick: Laplacian Operator*

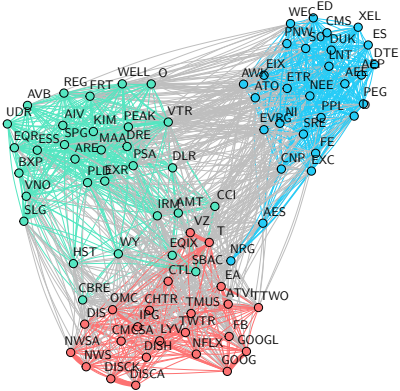
- $\boldsymbol{L}$ contains p^2 elements but the structural constraints reduce the effective degrees of freedom to $p(p-1)/2$. A more compact representation uses the vector $\boldsymbol{w} \in \mathbb{R}_+^{p(p-1)/2}$ of edge weights (off-diagonal upper-triangular entries of $\boldsymbol{L}$).
- The linear **Laplacian operator** $\mathcal{L}(\boldsymbol{w})$ reconstructs the full Laplacian — e.g., for a 3-node graph:

$$\mathcal{L}(\boldsymbol{w}) = \begin{bmatrix} w_1 + w_2 & -w_1 & -w_2 \\ -w_1 & w_1 + w_3 & -w_3 \\ -w_2 & -w_3 & w_2 + w_3 \end{bmatrix}.$$

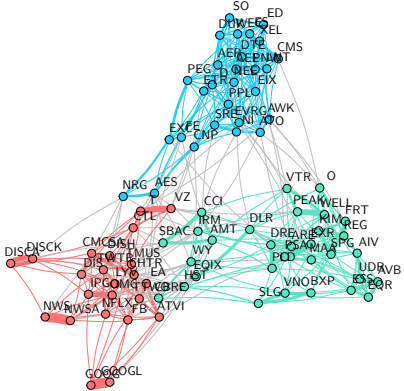
- $\boldsymbol{L} = \mathcal{L}(\boldsymbol{w})$ automatically satisfies all structural constraints $\boldsymbol{L}\mathbf{1} = \mathbf{0}$, $L_{ij} = L_{ji} \leq 0$, $\forall\, i \neq j$, so the optimization can be carried out directly over $\boldsymbol{w} \geq \mathbf{0}$ (Kumar et al. 2019, 2020).

Numerical Experiments: Effect of Sparsity Regularizer

S&P 500 stocks (3 sectors: Industrials, Consumer Staples, Energy; 2016–2019):



GMRF graph (ℓ_1-norm)



GMRF graph (reweighted ℓ_1-norm)

▶ *The ℓ_1-norm produces a dense graph — motivating the need for a concave (reweighted ℓ_1) regularizer.*

Outline

Graph Shift Operator and Graph Stationarity

- The **Graph Shift Operator (GSO)** $\boldsymbol{S} \in \mathbb{R}^{p \times p}$ encodes the graph topology: $S_{ij} \neq 0$ iff $i = j$ or nodes i and j are connected by an edge. Common choices: adjacency matrix $\boldsymbol{W}$ or Laplacian $\boldsymbol{L}$.
- A **graph filter** is a polynomial of the GSO: $\boldsymbol{H} = \sum_{k=0}^{K} h_k \boldsymbol{S}^k$. It operates locally — each node aggregates information from its K-hop neighborhood.
- **Physical interpretation:** since $\boldsymbol{S}^k$ propagates a signal k hops through the graph, $\boldsymbol{H}$ captures **multi-hop (delayed) interactions** among nodes — as opposed to an unstructured precision matrix $\boldsymbol{\Theta}$, which only models instantaneous pairwise interactions.
- A graph signal $\boldsymbol{x}$ is **stationary on** $\boldsymbol{S}$ if its covariance $\boldsymbol{\Sigma}$ commutes with $\boldsymbol{S}$: $\boldsymbol{\Sigma}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{\Sigma}$. Equivalently, $\boldsymbol{\Sigma}$ and $\boldsymbol{S}$ share the same eigenvectors, i.e., $\boldsymbol{\Sigma}$ is a polynomial of $\boldsymbol{S}$:

$$\boldsymbol{\Sigma} = \boldsymbol{H}^2 = \left(\sum_{k=0}^{K} h_k \boldsymbol{S}^k \right)^2 = p(\boldsymbol{S}).$$

The PGL Formulation

- **Recall:** the Graphical LASSO estimates the sparse precision matrix $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}$:

$$\underset{\boldsymbol{\Theta} \succ \mathbf{0}}{\text{maximize}} \quad \log\det(\boldsymbol{\Theta}) - \text{Tr}(\boldsymbol{\Theta}\hat{\boldsymbol{\Sigma}}^{\text{SCM}}) - \rho\|\boldsymbol{\Theta}\|_{1,\text{off}},$$

  where $\hat{\boldsymbol{\Sigma}}^{\text{SCM}}$ is the sample covariance matrix; but ignores any underlying graph structure.

- **PGL** jointly estimates $\boldsymbol{\Theta}$ and the GSO $\boldsymbol{S}$ under stationarity — the commutativity constraint $\boldsymbol{\Theta}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{\Theta}$ encodes that $\boldsymbol{\Theta}$ is a polynomial of $\boldsymbol{S}$ (Buciulea et al. 2025):

$$\begin{aligned} \underset{\boldsymbol{\Theta} \succ \mathbf{0},\, \boldsymbol{S} \in \mathcal{S}}{\text{maximize}} \quad & \log\det(\boldsymbol{\Theta}) - \text{Tr}(\hat{\boldsymbol{\Sigma}}^{\text{SCM}}\boldsymbol{\Theta}) - \rho\|\boldsymbol{S}\|_0 \\ \text{subject to} \quad & \boldsymbol{\Theta}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{\Theta}, \end{aligned}$$

  where $\mathcal{S}$ encodes structural constraints on $\boldsymbol{S}$ (symmetric, non-negative off-diagonal, zero diagonal, $\boldsymbol{S}\mathbf{1} \geq \mathbf{1}$).

PGL Algorithm

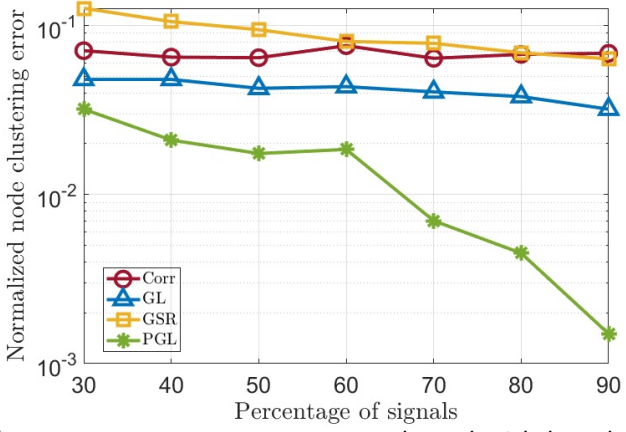
- **Two challenges:** the ℓ_0 sparsity penalty and the commutativity constraint $\boldsymbol{\Theta S} = \boldsymbol{S\Theta}$ are both non-convex.

- **Solution strategy:** relax ℓ_0 to a reweighted ℓ_1, then apply **Majorization-Minimization (MM)** (Sun, Babu, and Palomar 2017) — alternating between two closed-form updates (Buciulea et al. 2025):
  - Update $\boldsymbol{\Theta}$ with $\boldsymbol{S}$ fixed: $\boldsymbol{\Theta}^\star = \boldsymbol{V}\,\mathrm{diag}\Big(\frac{1}{2}\Big(-\hat{\boldsymbol{\lambda}} + \sqrt{\hat{\boldsymbol{\lambda}}^2 + 4}\Big)\Big)\,\boldsymbol{V}^{\mathsf{T}}$, where $\hat{\boldsymbol{\lambda}}$ are the eigenvalues of $\hat{\boldsymbol{\Sigma}}^{\mathrm{SCM}}$ in the basis of $\boldsymbol{S}$.
  - Update $\boldsymbol{S}$ with $\boldsymbol{\Theta}$ fixed: element-wise soft-thresholding in the eigenbasis of $\boldsymbol{\Theta}$.

- **Guarantees:** converges to a stationary point; a fast variant (GST-fast) reduces computational complexity significantly.

PGL Numerical Experiments: Stock Clustering

S&P 500 data (40 stocks, 4 sectors): clustering error vs. fraction of data used for graph learning (Buciulea et al. 2025):



▶ *PGL recovers the sector structure more accurately and with less data than GL, GSR, and correlation baselines.*

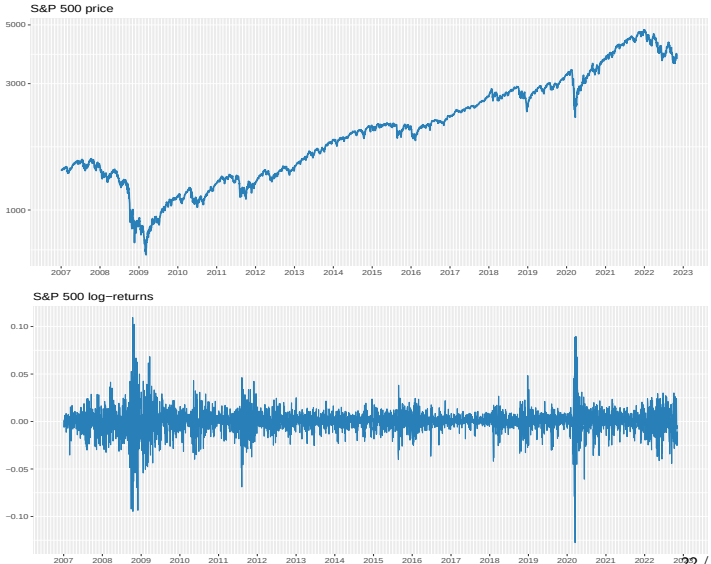
Outline

Prices and Returns

- **Price:** p_t (discrete time periods: minutes to years)
- **Log-price:** $y_t \triangleq \log p_t$
- **Linear return:** $r_t^{\text{lin}} \triangleq \dfrac{p_t - p_{t-1}}{p_{t-1}}$
- **Log-return:** $r_t \triangleq y_t - y_{t-1} = \log\left(\dfrac{p_t}{p_{t-1}}\right)$
- **Random walk model:**

$$y_t = \mu + y_{t-1} + \epsilon_t \Rightarrow r_t = \mu + \epsilon_t$$

(Ruppert 2010; Palomar 2025, ch.2)



Stylized Facts

Empirical regularities observed across instruments, markets, and time periods (Cont 2001):

Absence of Return Autocorrelations

- Returns are largely unpredictable (efficient-market hypothesis, Fama 1970)
- But $|r_t|$ and r_t^2 *are* autocorrelated — nonlinear dependence persists

Volatility Clustering / Lack of Stationarity

- Large moves tend to follow large moves; variance changes over time (Mandelbrot 1963)

$\Rightarrow$ *Addressed in Section 7: Learning Time-Varying Graphs*

Heavy Tails

- Distributions exhibit much heavier tails than Gaussian; more frequent extreme events

$\Rightarrow$ *Addressed in Section 5: Learning Heavy-Tailed Graphs*

Correlation Structure

- Assets co-move: pairwise correlations are predominantly positive and reveal market factor structure

$\Rightarrow$ *Addressed in Section 6: Learning Structured Graphs*

Stylized Fact 1: Heavy Tails

Gaussian distribution (fully characterized by mean and variance):

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

Skewness (asymmetry) and **kurtosis** (tail thickness):

$$\gamma_1 = \frac{\mathbb{E}[(r-\mu)^3]}{\sigma^3}, \qquad \kappa = \frac{\mathbb{E}[(r-\mu)^4]}{\sigma^4}$$

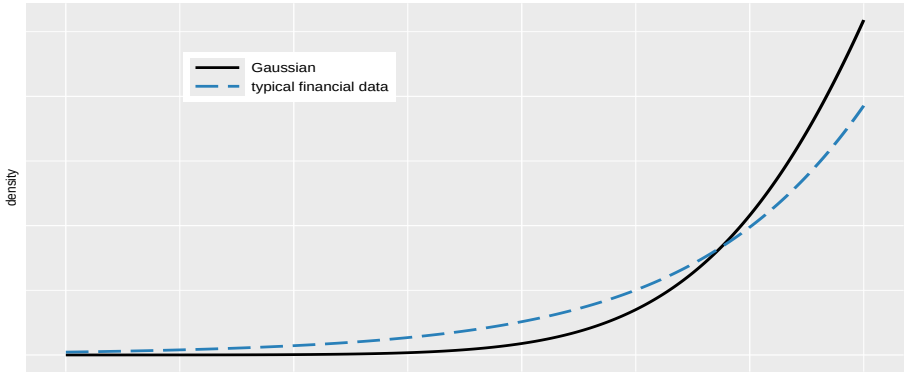
- Gaussian: $\gamma_1 = 0$, $\kappa = 3$ (excess kurtosis $= 0$)

Heavy tails: kurtosis > 3 — tail decay slower than Gaussian

- More frequent extreme events than Gaussian predicts
- Financial returns exhibit heavier tails, especially on the left (large losses)

(Ruppert 2010; Palomar 2025, ch.2)

Stylized Fact 1: Heavy Tails (cont.)

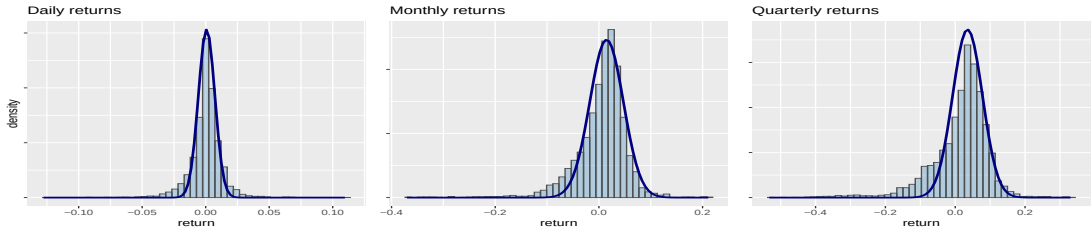


Implications for finance:

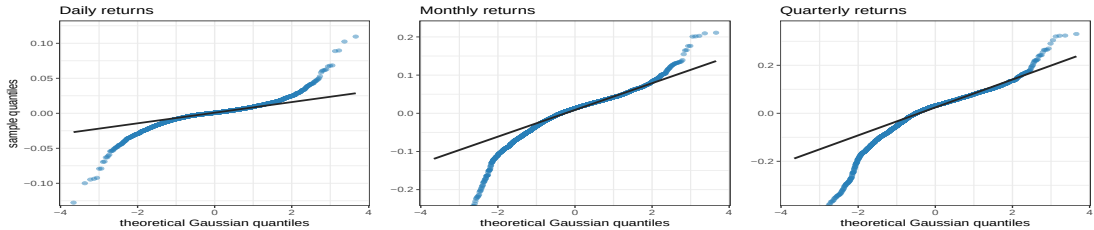
- **Risk underestimation**: Gaussian models underestimate the probability of crashes and large losses
- **Portfolio optimization**: ignoring heavy tails leads to portfolios that are too aggressive

Stylized Fact 1: Heavy Tails (S&P 500)

S&P 500 log-return histograms at increasing frequencies:



Q-Q plots vs. Gaussian (deviations from the line = heavy tails):



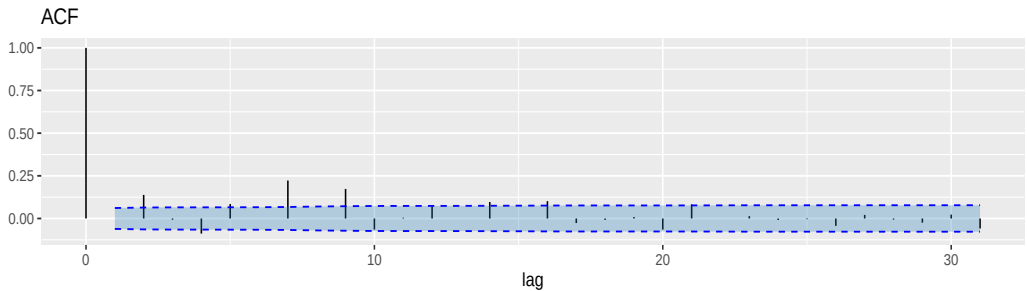
Stylized Fact 2: Volatility Clustering

Efficient-Market Hypothesis (EMH)

- Share prices reflect **all available information** at all times (Fama, 1970)
- Neither technical nor fundamental analysis can consistently generate excess returns
- In practice: financial returns behave approximately as **i.i.d.**.

Linear temporal structure: ACF of returns

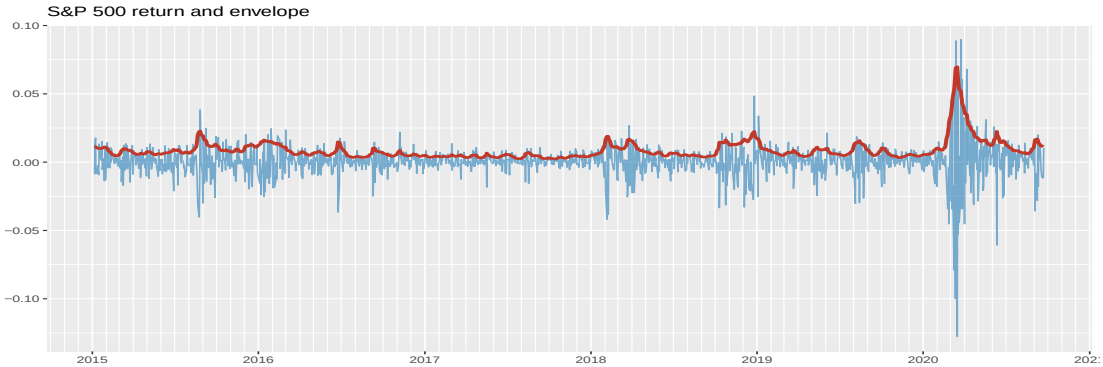
- ACF measures linear dependence between r_t and r_{t-k}
- EMH predicts no significant autocorrelation in returns
- Empirically confirmed for S&P 500: ACF lags lie within noise bands



Stylized Fact 2: Volatility Clustering (cont.)

Nonlinear structure: the volatility envelope clusters

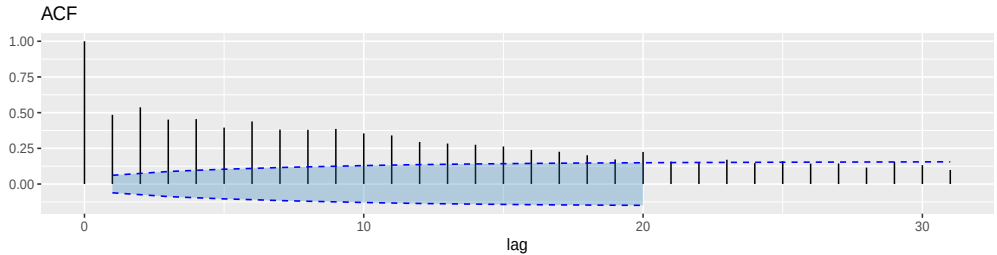
- Large changes tend to follow large changes (high volatility periods)
- Small changes tend to follow small changes (calm periods)
- Documented by Mandelbrot (1963) and Fama (1965)
- Modeled by ARCH/GARCH family of processes



Stylized Fact 2: Volatility Clustering (cont.)

ACF of $|r_t|$: nonlinear dependence persists

- Unlike ACF of r_t, the ACF of $|r_t|$ is significant at many lags
- Confirms that volatility — not returns — carries temporal structure



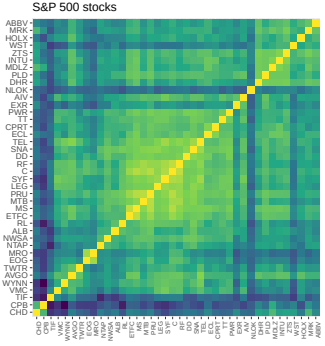
Implications for finance:

- **Volatility is forecastable**: ARCH/GARCH models exploit this for risk management
- **Non-stationarity**: the statistics and distribution of financial returns change over time — addressed later when learning time-varying graphs

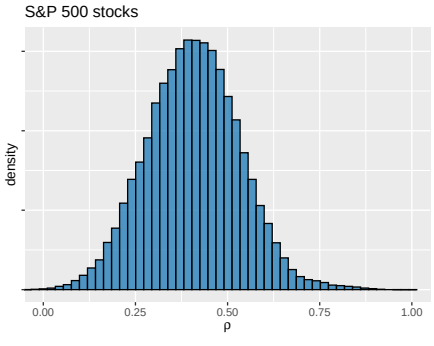
Stylized Fact 3: Correlation Structure

Assets are not independent — pairwise correlations are predominantly positive

- Stocks co-move: when the market rises (falls), most stocks rise (fall) together
- Ignoring correlations leads to severe underestimation of portfolio risk



Correlation matrix of S&P 500 stocks



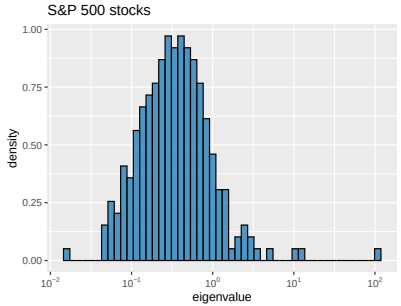
Distribution of pairwise correlations

(Cont 2001; Palomar 2025, ch.2)

Stylized Fact 3: Correlation Structure (cont.)

Factor model structure: one dominant eigenvalue

- Eigenvalues of the correlation matrix reveal the factor structure of the market
- One very large eigenvalue $\gg$ all others: the **market factor** drives most of the variance
- Remaining smaller eigenvalues carry asset-specific (idiosyncratic) structure



Implication: after removing the dominant market factor, the residual correlations reflect genuine pairwise relationships between stocks — naturally modeled as a **graph**

(Palomar 2025, ch.2)

Implications for Graph Learning

The three stylized facts jointly motivate the models developed in this talk:

- **Heavy tails**: the Gaussian distribution is inadequate — a **heavy-tailed distribution model** is needed for robust graph learning (and perhaps even asymmetric)
- **Volatility clustering**: returns are not i.i.d. over time — a **time-varying graph** model is required to track the evolving market structure
- **Correlation structure**: pairwise asset relationships are non-trivial and structured (positive, low-rank, and with assets naturally **clustered** into groups of high correlation) — a **graph** is a natural and interpretable representation (connecting later to structured graph learning)

$\Rightarrow$ We need graph learning algorithms that handle **heavy tails**, **clustered assets**, and **time variation**

Outline

Heavy-Tailed Graph Model

- Recall that the **Gaussian model** $f(\boldsymbol{x}) \propto |\boldsymbol{\Sigma}|^{-1/2} \exp\Big(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\Big)$ leads to the MLE problem for the graph Laplacian:

$$\begin{array}{ll} \underset{\boldsymbol{L}\succeq\boldsymbol{0}}{\text{maximize}} & \log \operatorname{gdet}(\boldsymbol{L}) - \operatorname{Tr}(\boldsymbol{L}\boldsymbol{S}) \\ \text{subject to} & \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0}, \quad L_{ij} = L_{ji} \leq 0, \quad \forall\, i \neq j, \end{array}$$

  where $\boldsymbol{S} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}^{(t)}(\boldsymbol{x}^{(t)})^{\mathsf{T}}$ is the sample covariance matrix.

- **Student-t model** $f(\boldsymbol{x}) \propto |\boldsymbol{\Sigma}|^{-1/2}\Big(1 + \frac{1}{\nu}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\Big)^{-(p+\nu)/2}$ with degrees of freedom ν (heavier tails as $\nu \downarrow$) leads to:

$$\begin{array}{ll} \underset{\boldsymbol{L}\succeq\boldsymbol{0}}{\text{maximize}} & \log \operatorname{gdet}(\boldsymbol{L}) - \dfrac{p+\nu}{T}\displaystyle\sum_{t=1}^{T} \log\Big(1 + \frac{1}{\nu}(\boldsymbol{x}^{(t)})^{\mathsf{T}}\boldsymbol{L}\,\boldsymbol{x}^{(t)}\Big) \\ \text{subject to} & \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0}, \quad L_{ij} = L_{ji} \leq 0, \quad \forall\, i \neq j. \end{array}$$

Majorization-Minimization (MM) Algorithm

- The heavy-tailed formulation is **nonconvex**. The **MM framework** (Sun, Babu, and Palomar 2017) upper-bounds each log term:

$$\log\Big(1+\tfrac{1}{\nu}(\boldsymbol{x}^{(t)})^\mathsf{T}\boldsymbol{L}\boldsymbol{x}^{(t)}\Big) \le \text{const} + \frac{(\boldsymbol{x}^{(t)})^\mathsf{T}\boldsymbol{L}\boldsymbol{x}^{(t)}}{\nu+(\boldsymbol{x}^{(t)})^\mathsf{T}\boldsymbol{L}_0\boldsymbol{x}^{(t)}}.$$

- This yields a sequence of **Gaussianized** problems ($k=1,2,\ldots$):

$$\begin{array}{ll}\underset{\boldsymbol{L}\succeq\boldsymbol{0}}{\text{maximize}} & \log\text{gdet}(\boldsymbol{L}) - \text{Tr}(\boldsymbol{L}\boldsymbol{S}^k)\\ \text{subject to} & \boldsymbol{L}\boldsymbol{1}=\boldsymbol{0}, \quad L_{ij}=L_{ji}\le 0, \quad \forall\, i\ne j,\end{array}$$

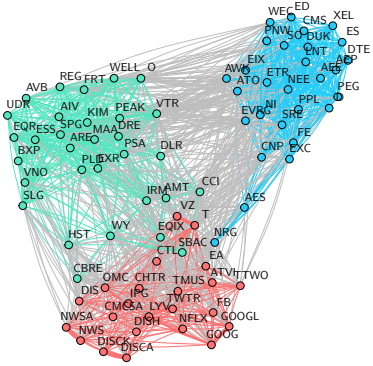
where $\boldsymbol{S}^k = \frac{1}{T}\sum_{t=1}^T w_t^k\,\boldsymbol{x}^{(t)}(\boldsymbol{x}^{(t)})^\mathsf{T}$ is a weighted sample covariance with weights (M. Cardoso, Ying, and Palomar 2021)

$$w_t^k = \frac{p+\nu}{\nu+(\boldsymbol{x}^{(t)})^\mathsf{T}\boldsymbol{L}^k\boldsymbol{x}^{(t)}}.$$

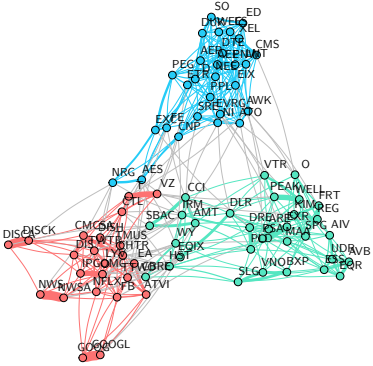
(Palomar 2025, ch. 5)

Numerical Experiments: Gaussian vs. Heavy-Tailed Graphs with Stocks

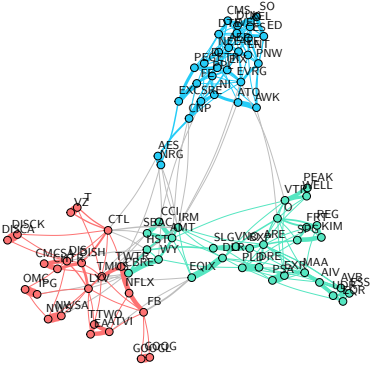
S&P 500 stocks (3 sectors: Industrials, Consumer Staples, Energy; 2016–2019):



GMRF graph (ℓ_1-norm)



GMRF graph (reweighted ℓ_1-norm)

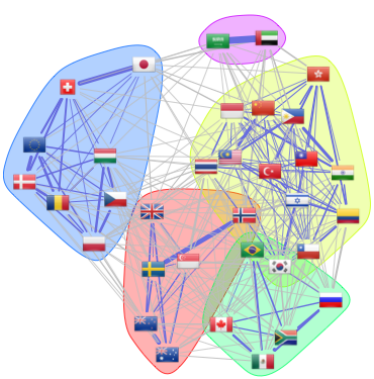


Heavy-tailed MRF graph

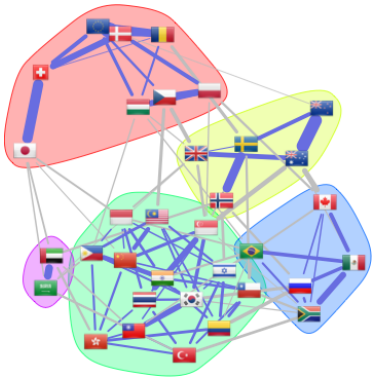
▶ *Heavy-tailed graphs better capture the actual structure of financial data — fewer spurious edges, cleaner sector clusters.*

Numerical Experiments: Gaussian vs. Heavy-Tailed Graphs with FX

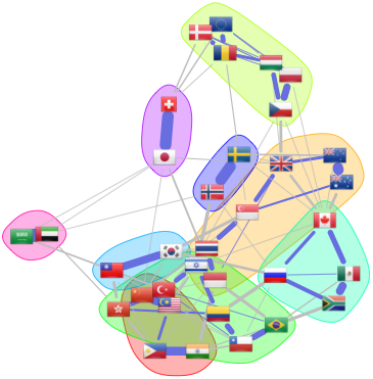
FX currencies (COVID period, 2020):



GMRF graph (ℓ_1-norm)



GMRF graph (reweighted ℓ_1-norm)

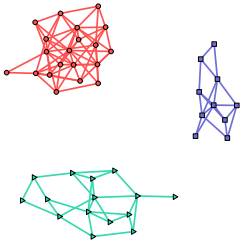


Heavy-tailed MRF graph

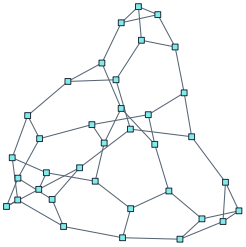
▶ *Heavy-tailed graphs highlight geographically close currency pairs (e.g., HKD/CNY, TWD/KRW, PLN/CZK) — invisible in Gaussian graphs.*

Outline

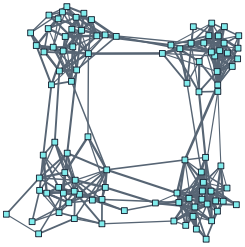
Structured Graphs



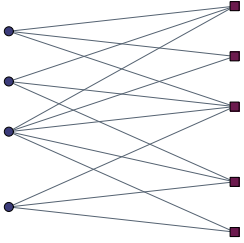
Multi-component



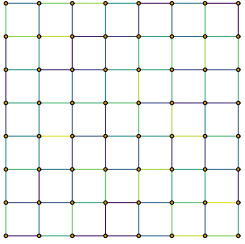
Regular



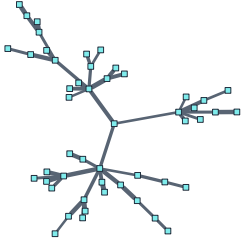
Modular



Bipartite



Grid



Tree

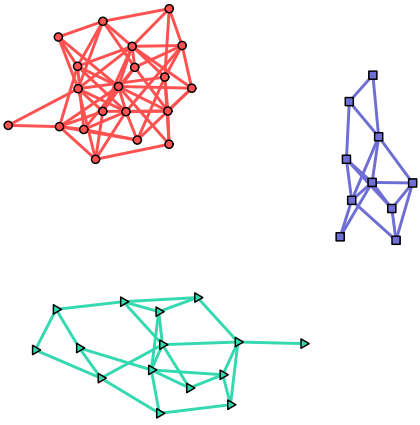
Learning Structured Graphs via Eigenvalues

- Learning graphs with a **specific structure** is generally NP-hard; no universal algorithm exists.
- Simple structures (regular, grid): enforced by fixing elements of $\boldsymbol{L}$ or controlling node degrees.
- Complex structures: characterized by **spectral properties** of $\boldsymbol{L}$ or $\boldsymbol{W}$ — enforced in the learning problem:
  - k-**component graph:** $\boldsymbol{L}$ has k zero eigenvalues (low-rank structure).
  - **Bipartite graph:** $\boldsymbol{W}$ has eigenvalues symmetric around zero.

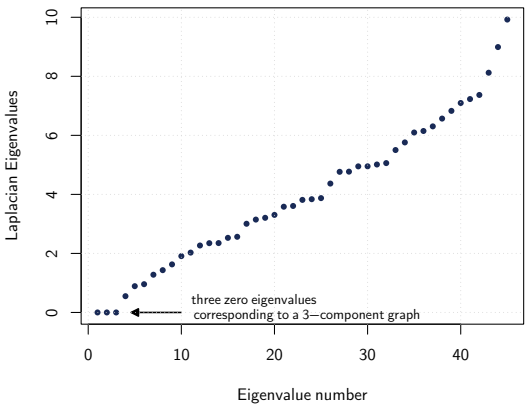
(Palomar 2025, ch. 5)

Example of k-Component Graph

3-component graph (3 clusters) along Laplacian matrix eigenvalues (3 zero eigenvalues):



Graph



Laplacian matrix eigenvalues

Learning k-Component Graphs

- Enforce low-rank structure via **Ky Fan's theorem**:

$$\sum_{i=1}^{k} \lambda_i(\boldsymbol{L}) = \min_{\boldsymbol{F} \in \mathbb{R}^{p \times k}:\, \boldsymbol{F}^\top \boldsymbol{F} = \boldsymbol{I}} \operatorname{Tr}(\boldsymbol{F}^\top \boldsymbol{L} \boldsymbol{F}),$$

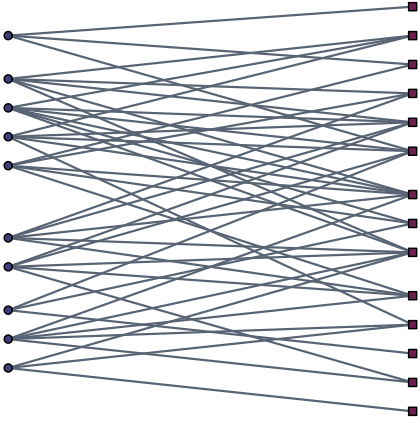
  so it suffices to add the penalty $\operatorname{Tr}(\boldsymbol{F}^\top \boldsymbol{L} \boldsymbol{F})$ to the objective.
- Combining with the heavy-tailed MM framework yields a sequence of Gaussianized problems ($k = 1, 2, \dots$) (Cardoso et al. 2021):

$$\begin{array}{ll} \underset{\boldsymbol{L} \succeq \boldsymbol{0},\, \boldsymbol{F}}{\text{maximize}} & \log \operatorname{gdet}(\boldsymbol{L}) - \operatorname{Tr}(\boldsymbol{L}\boldsymbol{S}^k) - \gamma \operatorname{Tr}(\boldsymbol{F}^\top \boldsymbol{L} \boldsymbol{F}) \\ \text{subject to} & \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0}, \quad L_{ij} = L_{ji} \leq 0, \quad \forall\, i \neq j, \\ & \boldsymbol{F}^\top \boldsymbol{F} = \boldsymbol{I}, \end{array}$$

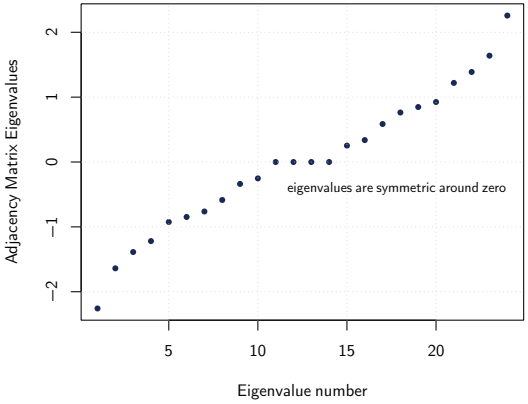
  where $\boldsymbol{S}^k$ is the weighted sample covariance from the MM iteration.
- Solved efficiently by alternating between updates of $\boldsymbol{L}$ (with $\boldsymbol{F}$ fixed) and $\boldsymbol{F}$ (eigenvectors of $\boldsymbol{L}$).

Example of Bipartite Graph

Bipartite graph along adjacency matrix eigenvalues (symmetric around zero):



Graph



Adjacency matrix eigenvalues

Learning Bipartite Graphs

- Two types of nodes with edges only between types — useful to learn connections between individual assets and sector indices.
- Enforce bipartite structure by constraining $\boldsymbol{L}$ to have the block form (with $\boldsymbol{B} \in \mathbb{R}_+^{r \times q}$ containing edge weights):

$$\boldsymbol{L} = \begin{bmatrix} \mathrm{Diag}(\boldsymbol{B}\mathbf{1}) & -\boldsymbol{B} \\ -\boldsymbol{B}^\mathsf{T} & \mathrm{Diag}(\boldsymbol{B}^\mathsf{T}\mathbf{1}) \end{bmatrix}.$$

- The heavy-tailed MM framework yields a sequence of Gaussianized problems ($k = 1, 2, \dots$) (Cardoso et al. 2022):

$$\begin{array}{ll} \underset{\boldsymbol{L} \succeq \mathbf{0},\, \boldsymbol{B}}{\text{maximize}} & \log \mathrm{gdet}(\boldsymbol{L}) - \mathrm{Tr}(\boldsymbol{L}\boldsymbol{S}^k) \\ \text{subject to} & \boldsymbol{L} = \begin{bmatrix} \mathrm{Diag}(\boldsymbol{B}\mathbf{1}) & -\boldsymbol{B} \\ -\boldsymbol{B}^\mathsf{T} & \mathrm{Diag}(\boldsymbol{B}^\mathsf{T}\mathbf{1}) \end{bmatrix}, \\ & \boldsymbol{B} \geq \mathbf{0}. \end{array}$$

Learning k-Component Bipartite Heavy-Tailed Graphs

- Combine the two previous structures: a k-component graph with bipartite connectivity.

- This leads to a sequence of Gaussianized problems ($k = 1, 2, \ldots$) with both $\boldsymbol{F}$ and $\boldsymbol{B}$ as additional variables (M. Cardoso, Ying, and Palomar 2022):

$$\begin{aligned} \underset{\boldsymbol{L} \succeq \boldsymbol{0},\, \boldsymbol{F},\, \boldsymbol{B}}{\text{maximize}} \quad & \log \operatorname{gdet}(\boldsymbol{L}) - \operatorname{Tr}(\boldsymbol{L}\boldsymbol{S}^k) - \gamma \operatorname{Tr}(\boldsymbol{F}^\mathsf{T}\boldsymbol{L}\boldsymbol{F}) \\ \text{subject to} \quad & \boldsymbol{L} = \begin{bmatrix} \operatorname{Diag}(\boldsymbol{B}\boldsymbol{1}) & -\boldsymbol{B} \\ -\boldsymbol{B}^\mathsf{T} & \operatorname{Diag}(\boldsymbol{B}^\mathsf{T}\boldsymbol{1}) \end{bmatrix}, \\ & \boldsymbol{B} \geq \boldsymbol{0}, \quad \boldsymbol{F}^\mathsf{T}\boldsymbol{F} = \boldsymbol{I}. \end{aligned}$$

Learning Graphs with Arbitrary Structures

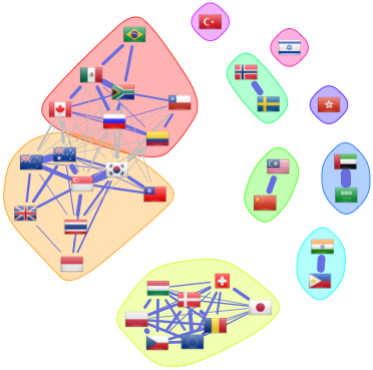
- Structural properties of many graph classes can be characterized by **spectral constraints**:
  - constraints on eigenvalues of the Laplacian matrix $\boldsymbol{L}$
  - constraints on eigenvalues of the adjacency matrix $\boldsymbol{W}$
- **General formulation** with spectral constraint (Kumar et al. 2019, 2020):

$$\begin{array}{ll} \underset{\boldsymbol{L}\succeq\boldsymbol{0}}{\text{maximize}} & \log \operatorname{gdet}(\boldsymbol{L}) - \operatorname{Tr}(\boldsymbol{L}\boldsymbol{S}) - \rho\|\boldsymbol{L}\|_{0,\text{off}} \\ \text{subject to} & \boldsymbol{L}\boldsymbol{1} = \boldsymbol{0}, \quad L_{ij} = L_{ji} \leq 0, \quad \forall\, i \neq j, \\ & \boldsymbol{\lambda}(\boldsymbol{L}) \in \mathcal{S}_{\boldsymbol{\lambda}}, \end{array}$$

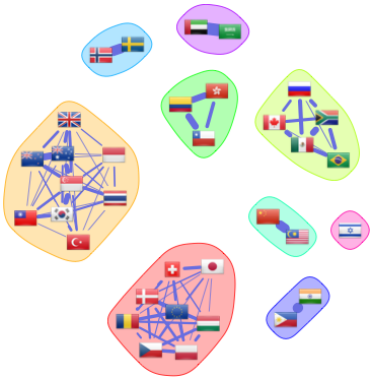
  where $\mathcal{S}_{\boldsymbol{\lambda}}$ encodes the desired spectral structure (e.g., k zero eigenvalues, symmetric eigenvalues).
- Nonconvex problem, but efficient algorithms exist based on MM and the Laplacian operator trick.

Numerical Experiments: Gaussian vs. Heavy-Tailed k-Component Graphs

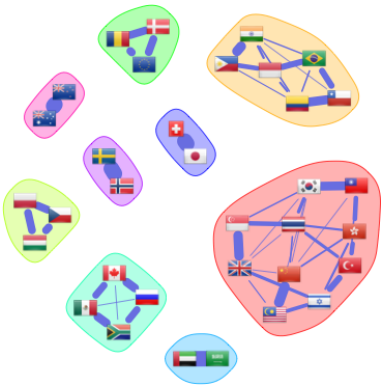
FX currencies (COVID period, 2020):



GMRF graph (ℓ_1-norm)



GMRF graph (reweighted ℓ_1-norm)

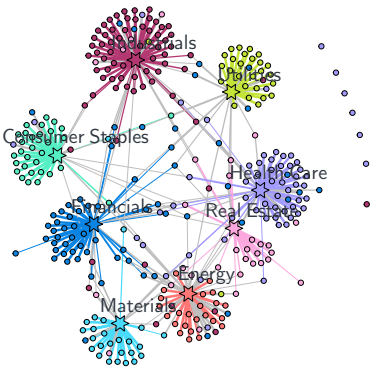


Heavy-tailed MRF graph

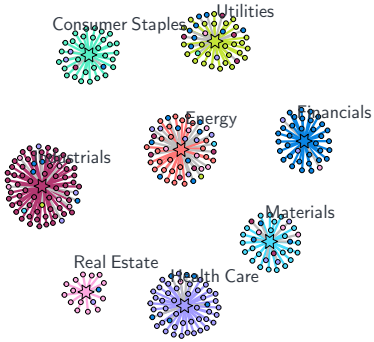
▶ *Heavy-tailed k-component graphs yield cleaner, more interpretable currency clusters than Gaussian counterparts.*

Numerical Experiments: Comparison of k-Component Bipartite Graphs

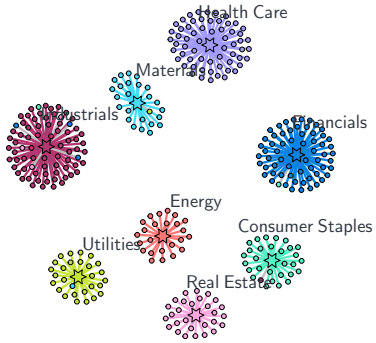
S&P 500 stocks (benchmarks: Kumar et al. 2020; Nie et al. 2017):



SGLA (Kumar et al., 2020)



SOBG (Nie et al., 2017)

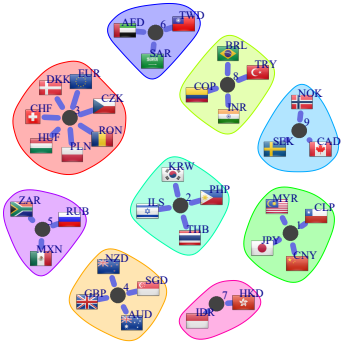


k-component bipartite heavy-tailed MRF

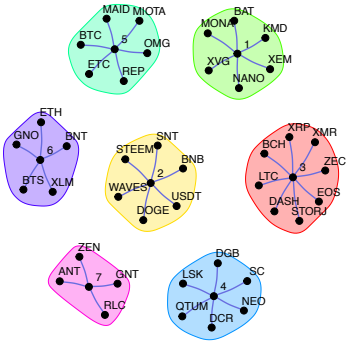
▶ *k-component bipartite heavy-tailed MRF graphs avoid isolated nodes and achieves better accuracy.*

Numerical Experiments: k-Component Bipartite Graphs with Unknown Indices

Applied to forex and crypto markets — unlike S&P 500, no predefined sector labels are available, so the k components and their cluster assignments are learned entirely from data (Javaheri and Palomar 2025).



Forex data



Crypto data

Outline

Motivation: Graphs Change Over Time

- Financial markets are **non-stationary**: correlations between assets shift during crises, regime changes, and macroeconomic events.

- **Rolling-window approach** (re-estimate graph on a sliding window of data):
  - Introduces estimation delay proportional to window size
  - Ignores temporal continuity — consecutive graphs are estimated independently
  - Not robust to outliers or missing observations

- **Principled time-varying formulation**: learn a sequence of graphs $\{\boldsymbol{L}_n\}$ jointly, coupling consecutive graphs through a temporal prior — capturing smooth evolution while adapting to regime shifts.

Time-Varying Graph Model

- Data arrives in **frames** $\mathcal{F}_n = \{t : t \in \text{frame } n\}$; each frame has its own graph $\mathcal{L}(\boldsymbol{w}_n)$ with edge weights $\boldsymbol{w}_n \in \mathbb{R}^{p(p-1)/2}$.
- The standard approach is to penalize the difference $\boldsymbol{w}_n - \boldsymbol{w}_{n-1}$ via the ℓ_2-norm or ℓ_1-norm.
- **Non-negative VAR(1) temporal evolution model**: edge weights evolve as

$$\boldsymbol{w}_n = (\boldsymbol{a} \odot \boldsymbol{w}_{n-1} + \varepsilon_n)_+, \quad n = 1, 2, \ldots$$

  where $\boldsymbol{a} \in \mathbb{R}_+^{p(p-1)/2}$ are VAR coefficients, ε_n is zero-mean noise, and $(\cdot)_+$ enforces non-negativity of edge weights.
- The **multiplicative factor** $\boldsymbol{a}$ better captures proportional changes in edge strengths over time.

(Javaheri et al. 2025)

Heavy Tails + Time-Varying: Combined Model

- **Laplacian operator** (Kumar et al. 2019, 2020): any non-negative edge weight vector $\boldsymbol{w} \in \mathbb{R}_+^{p(p-1)/2}$ defines a valid Laplacian via the linear mapping $\mathcal{L}(\boldsymbol{w})$: $\mathcal{L}(\boldsymbol{w})_{ij} = -w_{ij}$ for $i \neq j$ and $\mathcal{L}(\boldsymbol{w})_{ii} = \sum_{j \neq i} w_{ij}$. The optimization is then carried out directly over $\boldsymbol{w} \geq \mathbf{0}$.

- **Student-t likelihood** at each frame n combined with the VAR temporal prior yields (Javaheri et al. 2025):

$$\begin{aligned} \underset{\boldsymbol{w}_n \geq \mathbf{0},\, \boldsymbol{a} \geq \mathbf{0}}{\text{maximize}} \quad & \log \operatorname{gdet}(\mathcal{L}(\boldsymbol{w}_n)) - \frac{\nu + p}{T_n} \sum_{t \in \mathcal{F}_n} \log\left(1 + \frac{\boldsymbol{x}_t^\top \mathcal{L}(\boldsymbol{w}_n) \boldsymbol{x}_t}{\nu}\right) \\ & -\alpha \|\boldsymbol{w}_n - \boldsymbol{a} \odot \hat{\boldsymbol{w}}_{n-1}\|_1 - \beta \|\boldsymbol{w}_n\|_0 - \gamma \|\boldsymbol{a}\|_1 \\ \text{subject to} \quad & \boldsymbol{w}_n \in \Omega_w, \end{aligned}$$

where Ω_w encodes k-component structure and degree constraints.

Handling Corrupted and Missing Data

- In practice, financial data contains **missing observations and outlier-corrupted measurements**:

$$\boldsymbol{y}_t = \boldsymbol{m}_t \odot (\boldsymbol{x}_t + \boldsymbol{n}_t), \quad \boldsymbol{n}_t \sim \mathcal{N}(\boldsymbol{0}, \sigma_n^2 \boldsymbol{I}),$$

  where $\boldsymbol{m}_t \in \{0,1\}^p$ is a binary sampling mask ($m_{ti} = 0$ means asset i is missing at time t).

- The framework **jointly recovers the clean signal $\boldsymbol{x}_t$ and learns the graph $\boldsymbol{L}_{w_n}$** — missing entries are imputed as a byproduct.

- The MM weights $w_t^k \propto (\nu + \boldsymbol{x}_t^\top \boldsymbol{L}_{w_n} \boldsymbol{x}_t)^{-1}$ automatically **downweight outliers**: observations that are inconsistent with the graph structure receive lower weight.

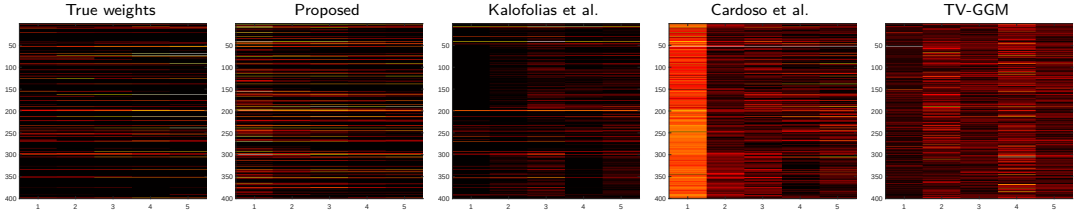
(Javaheri et al. 2025)

Algorithm: MM + ADMM

- The combined problem is non-convex. An **MM** outer loop linearizes the Student-t log terms, yielding a sequence of tractable subproblems solved via **ADMM**.
- Each iteration alternates five closed-form updates:
  - $\boldsymbol{L}_n$-update: eigendecomposition enforcing k-component rank constraint
  - $\boldsymbol{w}_n$-update: soft-thresholding enforcing sparsity (ℓ_0 relaxed to weighted ℓ_1)
  - $\boldsymbol{X}_n$-update: gradient step for signal recovery / missing-data imputation
  - $\boldsymbol{a}$-update: soft-thresholding for VAR coefficient learning
  - Dual variable update: standard ADMM multiplier ascent
- **Complexity:** $\mathcal{O}(p^3 + T_n p^2)$ per iteration; converges to a stationary point.
- The following slides show experiments on synthetic data and real S&P 500 market data (Javaheri et al. 2025).

Numerical Experiments: Synthetic Data

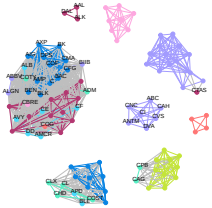
Recovered edge weights over time (darker = stronger edge):



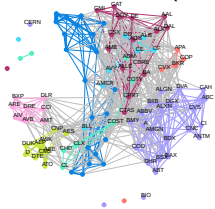
- *Proposed method recovers the ground-truth edge weight evolution most faithfully.*

Numerical Experiments: k-Component Graphs (S&P 500)

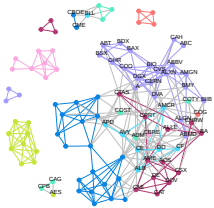
100 S&P 500 stocks partitioned into $k = 8$ sectors (last frame):



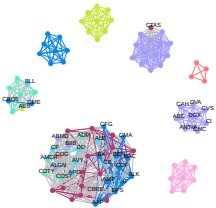
CLR (ACC 0.56)



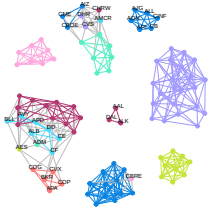
SGLA (ACC 0.27)



Fingraph (ACC 0.49)



Javaheri et al. (ACC 0.61)

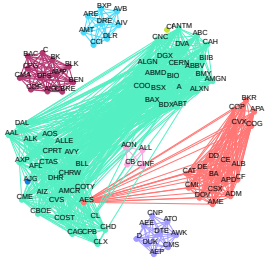


Proposed (ACC 0.69)

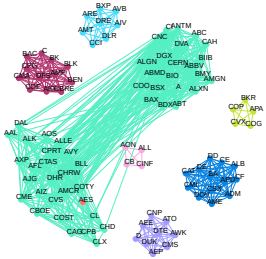
- *Proposed method cleanly separates sectors with fewer misclassified stocks.*

Numerical Experiments: Time Evolution of Graphs

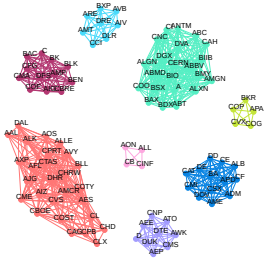
Graph structure evolves across three consecutive frames (S&P 500):



Frame 1



Frame 2

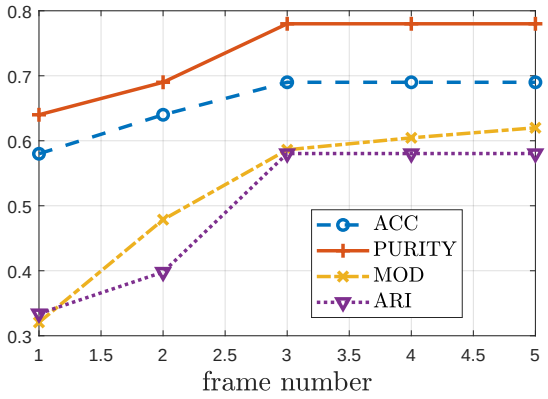


Frame 3

- *Cluster structure remains stable while edge strengths adapt to changing markets.*

Numerical Experiments: Clustering Accuracy Over Time

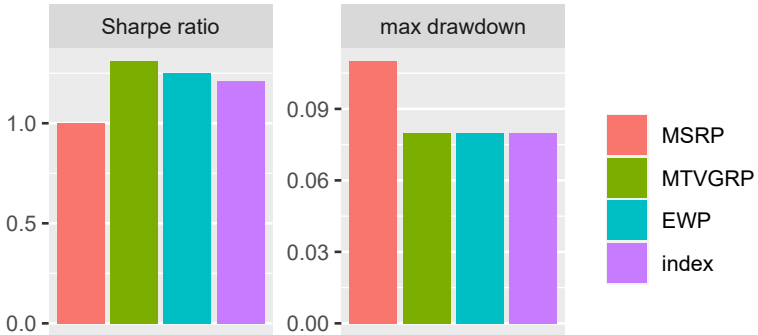
Clustering performance improves as more frames are observed (S&P 500, $k = 8$):



- *Time-varying model accumulates information across frames, steadily improving accuracy.*

Numerical Experiments: Portfolio Performance

Portfolio backtesting on S&P 500 (Sharpe ratio and maximum drawdown):



▶ *Time-varying graph portfolio (MTVGRP) achieves higher Sharpe ratio (1.31 vs. 1.00) and lower drawdown than benchmarks.*

Outline

Summary

- **Financial data is non-Gaussian and time-varying:** heavy tails, volatility clustering, and correlation structure all call for specialized graph models
- **Graph learning from smooth signals / GMRF** (Sections 2–3): a rich family of formulations — sparse, structured, and polynomial — for static graphs
- **Polynomial Graphical Lasso** (Section 3): exploits graph stationarity ($\boldsymbol{\Sigma S} = \boldsymbol{S\Sigma}$) to jointly learn the precision matrix and the graph, capturing multi-hop interactions (Buciulea et al. 2025)
- **Heavy-tailed and structured graphs** (Sections 5–6): Student-t model via MM yields robust estimation; spectral constraints enforce k-component and bipartite structure (M. Cardoso, Ying, and Palomar 2021, 2022; Kumar et al. 2020)
- **Time-varying graphs** (Section 7): non-negative VAR(1) temporal prior + Student-t likelihood tracks market dynamics, detects crises, and improves portfolio Sharpe ratio (Javaheri et al. 2025)

Thanks

For more information visit:

https://portfoliooptimizationbook.com
https://www.danielppalomar.com
https://github.com/convexfi
https://www.youtube.com/danielpalomar

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