Vertex-frequency Hypergraph Signal Processing: Analytic Tools and Applications

Alcebiades Dal Col

Federal University of Espirito Santo (Brazil)

Coauthors

Fabiano Petronetto, José R. de Oliveira Neto, Juliano B. Lima

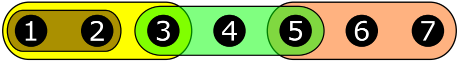
The Fourier transform has recently gained some versions in the Hypergraph Signal Processing scenario.

In a previous work [Dal Col et al., 2024], we introduced a Hypergraph Fourier Transform (HGFT), which allows us to generalize the windowed Fourier transform to hypergraphs.

In this presentation, we demonstrate how other vertex-frequency analysis tools can be extended to hypergraphs using our HGFT.

Hypergraph overview

A *hypergraph* $H = (V(H), E(H))$ is composed of $V(H) = \{v_1, v_2, \ldots, v_N\}$ and $E(H) = \{e_1, e_2, \ldots, e_K\}$.



The *maximum cardinality of the hyperedges* M is $M = \max\{|e|; e \in E(H)\}$ [Zhang et al., 2019, Pena-Pena et al., 2023].

The *adjacency tensor* $\mathcal{A} = [a_{i_1 i_2 \ldots i_M}]$ of a hypergraph H is a tensor defined based on the hyperedges of H [Dal Col et al., 2025].

The *degree tensor* $\mathcal{D} = [d_{i_1 i_2 \dots i_M}]$ is a super-diagonal tensor with diagonal entries

$$d_{\underbrace{ii\dots i}_{M\text{times}}} = \sum_{i_2, i_3, \dots, i_M = 1}^{N} a_{i i_2 i_3 \dots i_M}. \tag{1}$$

The *Laplacian tensor* $\mathcal{L}$ is the tensor $\mathcal{L} = \mathcal{D} - \mathcal{A}$.

Given a tensor $\mathcal{A} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$, its *frontal slices* are the matrices

$$A^{(k)} := \mathcal{A}(:,:,k) \in \mathbb{R}^{N_1 \times N_2}, \quad k = 1, 2, \ldots, N_3. \tag{2}$$

The *t-eigendecomposition* of a symmetric tensor $\mathcal{A} \in \mathbb{R}^{N_1 \times N_1 \times N_3}$ is

$$\mathcal{A} = \mathcal{V} * \Lambda * \mathcal{V}^T, \tag{3}$$

where Λ is an *f-diagonal tensor* ($\Lambda^{(k)}$, $k = 1, \ldots, N_3$, are diagonal matrices) and $\mathcal{V}$ is an *orthogonal tensor* ($\mathcal{V} * \mathcal{V}^T = \mathcal{I}$).

A symmetric tensor $\mathcal{A}$ can be t-decomposed if $\hat{A}^{(k)} = \hat{V}^{(k)} \hat{\Lambda}^{(k)} \hat{V}^{(k)^T}$ for $k = 1, 2, \ldots, N_3$ [Kilmer et al., 2013].

Given a hypergraph H with N vertices and $M = 3$, let

$\mathcal{L} \in \mathbb{R}^{N\times N\times N}$ be its Laplacian tensor,

$\mathcal{L}_s = \mathrm{sym}(\mathcal{L}) \in \mathbb{R}^{N\times N\times(2N+1))}$ the symmetrized version of $\mathcal{L}$

$\hat{\mathcal{L}}_s = \mathrm{fft}(\mathcal{L}_s, 3)$ the FFT of $\mathcal{L}_s$ along the third dimension, and

$\mathcal{L}_s = \mathcal{V} * \Lambda * \mathcal{V}^T$ the t-eigendecomposition of $\mathcal{L}_s$.

Since $\hat{L}_s^{(k)} = \hat{V}^{(k)}\hat{\Lambda}^{(k)}(\hat{V}^{(K)})^T$, $k = 1, 2, \ldots, N_s$ [Dal Col et al., 2025], the t-eigendecomposition of $\mathcal{L}_s$ exists.

A *hypergraph signal* is a function $x : V(H) \to \mathbb{R}$ defined on the vertices of the hypergraph $H = (V(H), E(H))$.

Hypergraph Fourier transform [Dal Col et al., 2024]

Let $x : V(H) \to \mathbb{R}$ be a hypergraph signal, its *hypergraph Fourier transform* (HGFT) is given by:

$$\hat{x}(\ell) := \langle x, \hat{V}^{(1)}(:, \ell) \rangle = \sum_{n=1}^{N} x(n) \hat{V}^{(1)}(n, \ell), \tag{4}$$

where $\hat{V}^{(1)}(:, \ell)$ is the ℓ-th column of the matrix $\hat{V}^{(1)}$.

Windowed hypergraph Fourier transform

A *windowed hypergraph Fourier atom* $y_{i,k} : V(H) \to \mathbb{R}$ is generated by the translation of a *window* $y : V(H) \to \mathbb{R}$ to the vertex v_i followed by the modulation by a scaling factor k,

$$y_{i,k}(n) := (M_k T_i y)(n) = N \hat{V}^{(1)}(n,k) \sum_{\ell=1}^{N} \hat{y}(\ell) \hat{V}^{(1)}(i,\ell) \hat{V}^{(1)}(n,\ell). \quad (5)$$

Windowed hypergraph Fourier transform [Dal Col et al., 2024]

The *windowed hypergraph Fourier transform* (WHGFT) of a hypergraph signal $x : V(H) \to \mathbb{R}$ is the projection of the signal onto the atoms,

$$WHGFT(x)(i,k) := \langle x, y_{i,k} \rangle = \sum_{n=1}^{N} x(n) y_{i,k}(n). \quad (6)$$

Spectral hypergraph wavelet transform

Let $\hat{y}_p : [0, \hat{\Lambda}^{(1)}(N,N)] \to \mathbb{R}$, $p = 1, \ldots, P$, be a dictionary of band-pass kernels, where $\hat{\Lambda}^{(1)}(N,N)$ is the largest eigenvalue of matrix $\hat{L}_s^{(1)}$.

The *scaled hypergraph wavelets* $\psi_{i,p} : V \to \mathbb{R}$, $i = 1, \ldots, N$, $p = 1, \ldots, P$, are defined by

$$\psi_{i,p}(n) = \sum_{\ell=1}^{N} \hat{y}_p \left(\hat{\Lambda}^{(1)}(\ell,\ell) \right) \hat{V}^{(1)}(i,\ell) \hat{V}^{(1)}(n,\ell). \tag{7}$$

A dictionary of band-pass kernels for graphs can be interpreted as a dictionary of band-pass kernels for hypergraphs as long as the eigenvalues of the graph Laplacian λ_ℓ are properly replaced by $\hat{\Lambda}^{(1)}(\ell,\ell)$.

Spectral hypergraph wavelet transform [Dal Col et al., 2025]

The *spectral hypergraph wavelet transform* (SHGWT) of a signal $x : V(H) \to \mathbb{R}$ is defined by

$$SGWT(f)(i,p) := \langle x, \psi_{i,p} \rangle = \sum_{\ell=1}^{N} \hat{y}_p \left(\hat{\Lambda}^{(1)}(\ell,\ell) \right) \hat{x}(\ell) \hat{V}^{(1)}(i,\ell), \qquad (8)$$

where $\hat{x}$ is the HGFT (4) of the hypergraph signal x.

A 3-*uniform (loose) path hypergraph* with N vertices is a 3-uniform hypergraph $H = (V(H), E(H))$, where $V(H) = \{v_1, v_2, \ldots, v_N\}$ and $E(H) = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \ldots, \{v_{N-2}, v_{N-1}, v_N\}\}$.

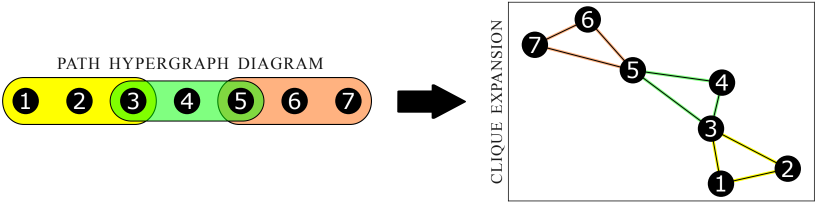


Figure: Path hypergraph with $N = 7$ vertices and its graph representation.

We generate a signal $\bar{x} : V(H) \to \mathbb{R}$ by combining three columns of $\hat{V}^{(1)}$

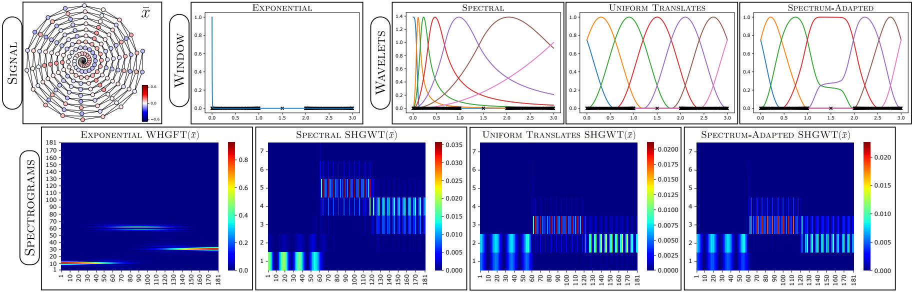


Figure: WHGFT with exponential window and SHGWT with spectral, uniform translates, and spectrum-adapted wavelets.

Hypergraph Tikhonov Regularization

Hypergraph Tikhonov regularization [Dal Col et al., 2025]

Let $x = x_0 + \eta$ be a signal, where x_0 is smooth and η is noise. We aim to recover the signal x_0 by solving the following optimization problem, called *hypergraph Tikhonov regularization*,

$$x_s = \min_{z}\{||z - x||^2 + \gamma z^T \hat{L}_s^{(1)} z\}. \tag{9}$$

Minimization of the *hypergraph quadratic form* [Dal Col et al., 2025]

$$x_s^T \hat{L}_s^{(1)} x_s = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\left(\sum_{k=1}^{N} a_{ijk}\right)(x_s(i) - x_s(j))^2\right) \tag{10}$$

ensures that x_s is smooth in relation to the hypergraph.

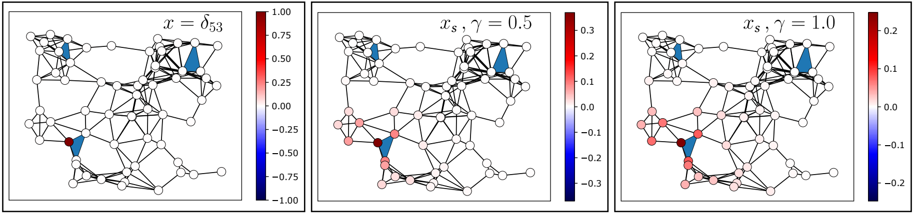


Figure: Regularization for a delta signal on a random geometric hypergraph

Hypergraph regularization centrality

Hypergraph regularization centrality [Dal Col et al., 2025]

Let $\delta_k : V \to \mathbb{R}$ be the delta signal centered in node v_k and $s_k : V \to \mathbb{R}$ its hypergraph Tikhonov regularization (9). The *hypergraph regularization centrality* (HGRC) of the node v_k is defined by

$$HGRC(v_k) = \frac{1}{s_k(v_k)}. \tag{11}$$

In the analysis, we consider two other centrality measures for hypergraphs inspired by degree and eigenvector centrality for graphs.

Hypergraph degree centrality

The *hypergraph degree centrality* of a node v_k is the k-th entry of the diagonal tensor $\mathcal{D}$.

Hypergraph eigenvector centrality

The *hypergraph eigenvector centrality* of a node v_k is $\hat{V}^{(1)}(k, N)$.

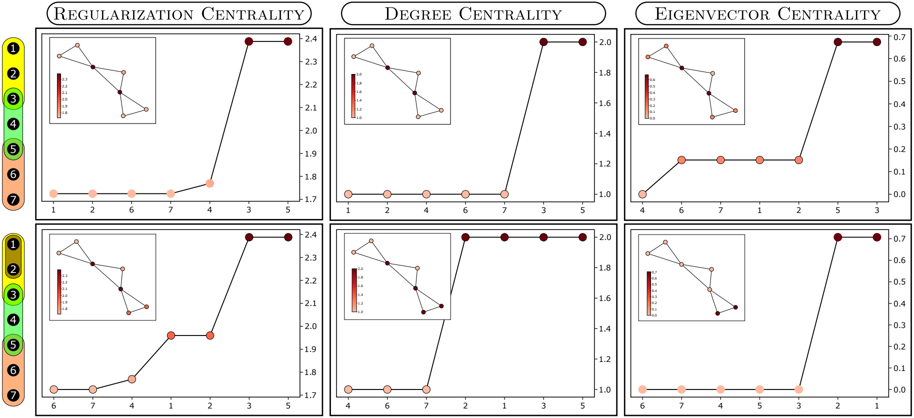


Figure: Hypergraph centrality measures for a path hypergraph with $N = 7$ vertices and for a path hypergraph with $N = 7$ vertices and additional hyperedge $\{v_1, v_2\}$.

Take-home message

This paper has extended the frontier of hypergraph signal processing by introducing novel vertex-frequency analysis tools.

As future work, we intend to explore the proposed tools in several potential applications.

References I

Dal Col, A., Petronetto, F., de Oliveira Neto, J. R., and Lima, J. B. (2024).
Windowed hypergraph fourier transform and vertex-frequency representation.
Signal Processing, 223:109538.

Dal Col, A., Petronetto, F., de Oliveira Neto, J. R., and Lima, J. B. (2025).
Vertex-frequency hypergraph signal processing: Analytic tools and applications.
Signal Processing, page 110277.

References II

Hammond, D. K., Vandergheynst, P., and Gribonval, R. (2018).
The spectral graph wavelet transform: Fundamental theory and fast computation.
In *Vertex-Frequency Analysis of Graph Signals*, pages 141–175. Springer.

Kilmer, M. E., Braman, K., Hao, N., and Hoover, R. C. (2013).
Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging.
SIAM Journal on Matrix Analysis and Applications, 34(1):148–172.

Pena-Pena, K., Lau, D. L., and Arce, G. R. (2023).
t-hgsp: Hypergraph signal processing using t-product tensor decompositions.
IEEE Transactions on Signal and Information Processing over Networks.

References III

Schaub, M. T., Zhu, Y., Seby, J.-B., Roddenberry, T. M., and Segarra, S. (2021).
Signal processing on higher-order networks: Livin' on the edge... and beyond.
Signal Processing, 187:108149.

Shuman, D. I., Wiesmeyr, C., Holighaus, N., and Vandergheynst, P. (2015).
Spectrum-adapted tight graph wavelet and vertex-frequency frames.
IEEE Transactions on Signal Processing, 63(16):4223–4235.

Zhang, S., Ding, Z., and Cui, S. (2019).
Introducing hypergraph signal processing: Theoretical foundation and practical applications.
IEEE Internet of Things Journal, 7(1):639–660.

Thank you very much!

alcebiades.col@ufes.br

[Dal Col et al., 2025]

codes

UNIVERSIDADE FEDERAL DO ESPÍRITO SANTO

VIRTUS IMPAVIDA UFPE

CAV

CNPq

FACEPE Fundação de Amparo à Ciência e Tecnologia do Estado de Pernambuco

The *adjacency tensor* $\mathcal{A} = [a_{i_1 i_2 \ldots i_M}]$ of a hypergraph H is a tensor defined based on the hyperedges of H [Dal Col et al., 2024].

A hyperedge $e = \{v_{l_1}, \ldots, v_{l_c}\}$ is encoded by the entries $a_{i_1 i_2 \ldots i_M}$, where c elements of $\{i_1, \ldots, i_M\}$ are exactly the same as $\{l_1, \ldots, l_c\}$ and $M - c$ elements are chosen from $\{l_1, \ldots, l_c\}$. Furthermore,

$$a_{i_1 i_2 \ldots i_M} = \frac{c}{\alpha}$$

with

$$\alpha = \sum_{\substack{k_1, k_2, \ldots, k_c \geq 1, \\ k_1 + k_2 + \ldots + k_c = M}} \frac{M!}{k_1! \cdot k_2! \cdot \ldots \cdot k_c!}.$$

Each term of the sum in α represents the number of possible permutations after choosing $M - c$ elements from $\{l_1, \ldots, l_c\}$.

The *t-product* of two tensors $\mathcal{A} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and $\mathcal{B} \in \mathbb{R}^{N_2 \times N_4 \times N_3}$ is a tensor $\mathcal{C} = \mathcal{A} * \mathcal{B} \in \mathbb{R}^{N_1 \times N_4 \times N_3}$ given by

$$\mathcal{C} = \text{fold}\left(\begin{bmatrix} A^{(1)} & A^{(N_3)} & \dots & A^{(2)} \\ A^{(2)} & A^{(1)} & \dots & A^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ A^{(N_3)} & A^{(N_3-1)} & \dots & A^{(1)} \end{bmatrix} \cdot \begin{bmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(N_3)} \end{bmatrix} \right), \tag{12}$$

where the *fold operator* stacks frontal slices.

The *symmetrized version* of a tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times N}$ is a tensor

$$\mathcal{A}_s = \mathrm{sym}(\mathcal{A}) = \mathrm{fold}\left(\begin{bmatrix} 0 \\ \frac{1}{2}A^{(1)} \\ \frac{1}{2}A^{(2)} \\ \vdots \\ \frac{1}{2}A^{(2)} \\ \frac{1}{2}A^{(1)} \end{bmatrix}\right) \in \mathbb{R}^{N \times N \times (2N+1)}. \tag{13}$$

The *clique expansion* [Schaub et al., 2021] of a hypergraph $H = (V(H), E(H))$ is a graph representation $G = (V(G), E(G), A)$, with $V(G) := V(H)$,

$$E(G) := \{\{v_i, v_j\} \mid v_i, v_j \in e, e \in E(H), v_i \neq v_j\}. \quad (14)$$

In other words, set $E(G)$ is composed of all 2-element subsets of any edge $e \in E(H)$, and $A = [a_{ij}]$ is a matrix where a_{ij} quantifies how many times the edge $\{v_i, v_j\}$ is derived from $E(H)$.

Consider a 3-uniform path hypergraph H with $N = 7$ vertices and an additional hyperedge of cardinality $c = 2$. More precisely, $H = (V(H), E(H))$ with $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and

$$E(H) = \{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_7\}, \{v_1, v_2\}\}.$$

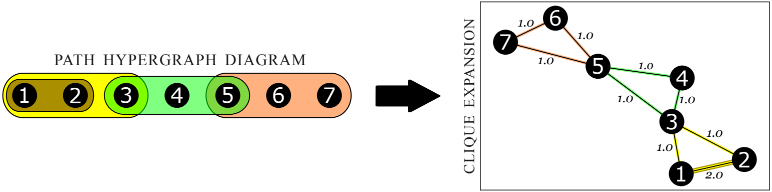


Figure: Path hypergraph with $N = 7$ vertices and an additional hyperedge and its clique expansion.

$E(G) = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\} \cup \{\{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\} \cup \{\{v_5, v_6\}, \{v_5, v_7\}, \{v_6, v_7\}\} \cup \{v_1, v_2\}$.

The Laplacian matrix of G is

$$L = \begin{bmatrix} 3.0 & -2.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -2.0 & 3.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & -1.0 & 4.0 & -1.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -1.0 & 2.0 & -1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -1.0 & -1.0 & 4.0 & -1.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & 2.0 & -1.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -1.0 & -1.0 & 2.0 \end{bmatrix}.$$

The first front slice of the tensor $\hat{\mathcal{L}}_s$ is given by

$$\hat{L}_s^{(1)} = \begin{bmatrix} 1.6667 & -1.1667 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ -1.1667 & 1.6667 & -0.5 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.5 & -0.5 & 2.0 & -0.5 & -0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.5 & 1.0 & -0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.5 & -0.5 & 2.0 & -0.5 & -0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & 1.0 & -0.5 \\ 0.0 & 0.0 & 0.0 & 0.0 & -0.5 & -0.5 & 1.0 \end{bmatrix}.$$

We generate a signal $\bar{x} : V(H) \to \mathbb{R}$ by combining three columns of $\hat{V}^{(1)}$

$$\bar{x}(n) = \begin{cases} \hat{V}^{(1)}(n, 11), & 1 \leq n \leq 60 \\ \hat{V}^{(1)}(n, 61), & 61 \leq n \leq 120 \\ \hat{V}^{(1)}(n, 31), & 121 \leq n \leq 181. \end{cases}$$

On graph representation, v_1 is at the center of the spiral and v_{181} is at the outer part.

We generate a signal $\bar{x} : V(H) \to \mathbb{R}$ by combining three columns of $\hat{V}^{(1)}$

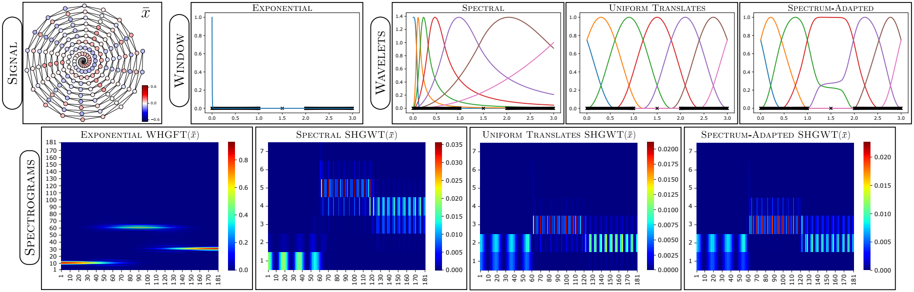


Figure: WHGFT with exponential window and SHGWT with spectral, uniform translates, and spectrum-adapted wavelets.

spectral *graph wavelets* [Hammond et al., 2018]
uniform translates *graph wavelets* [Shuman et al., 2015]
spectrum-adapted *graph* ***wavelets*** [Shuman et al., 2015]

Theorem

The solution to the optimization problem (9) *is given by*

$$x_s(n) = \sum_{\ell=1}^{N} \left[\frac{1}{1 + \gamma \hat{\Lambda}^{(1)}(\ell, \ell)} \right] \hat{x}(\ell) \hat{V}^{(1)}(n, \ell), \tag{15}$$

where $\hat{x}$ *is the hypergraph Fourier transform of* x.

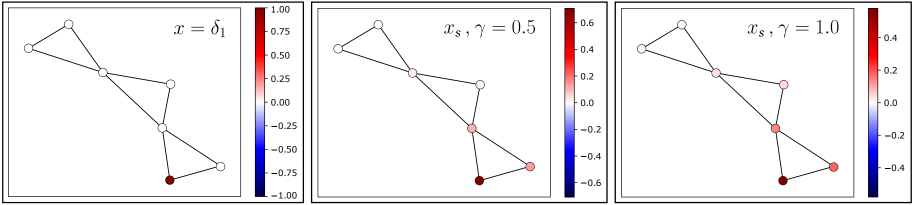


Figure: Regularization for the signal δ_1 on a path hypergraph

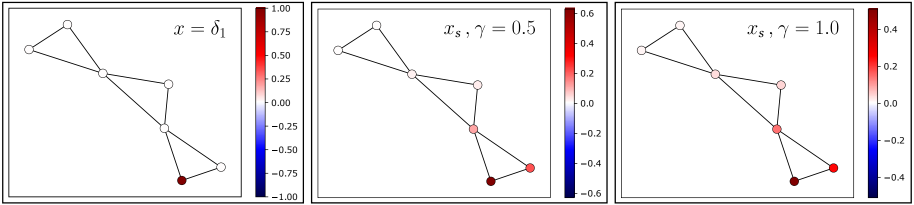


Figure: Regularization for the signal δ_1 on a path hypergraph with additional hyperedge $\{v_1, v_2\}$.

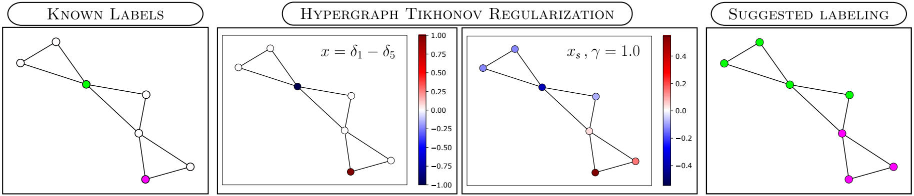


Figure: Classification for a path hypergraph with $N = 7$ vertices.