Optimal Wiener-Filter Solutions for Denoising Signals on Directed Graphs

Michael Chan, Alexandre Cionca, Dimitri Van De Ville

École Polytechnique Féderale de Lausanne,

Department of Electrical Engineering and Neuro-X Institute

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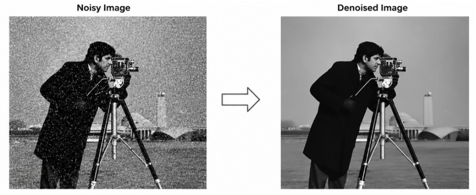
Outline

- Motivations
- (Graph) Wide Sense Stationary Signals
- (Graph) White Noise
- Optimal Graph Filter
- Preliminary Experiments

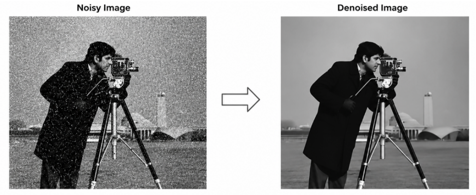
Motivations: Denoising Signals on Graphs



(a) Image denoising

- Why denoising signals?

Motivations: Denoising Signals on Graphs



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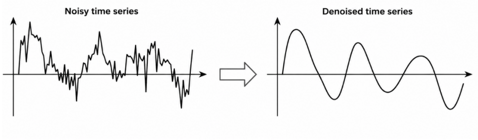

(b) Time-series denoising

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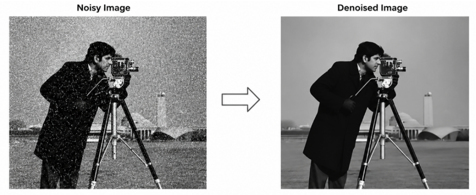

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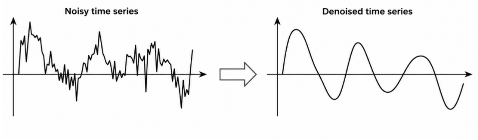

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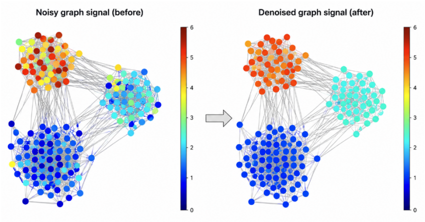

(c) Graph signal denoising.

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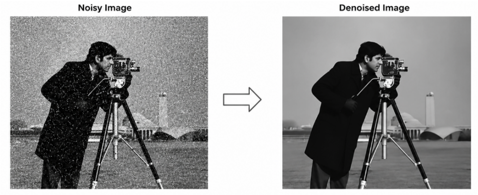

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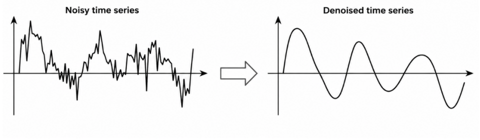

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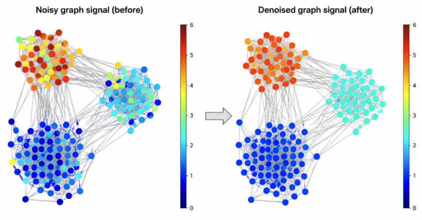

(c) Graph signal denoising.

- Why denoising signals?
- Leveraging domain structure to remove noise.

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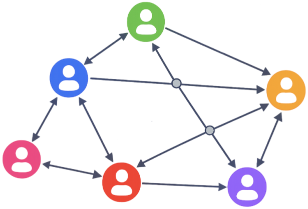

(a) Social network

- Applications in various cases (e.g., social networks, sensor networks, brain connectomics).

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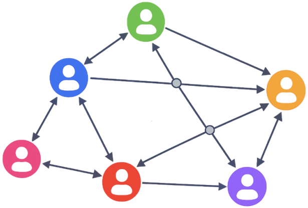

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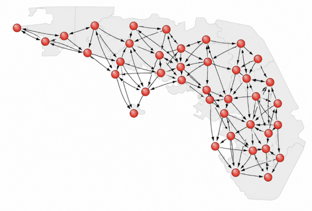

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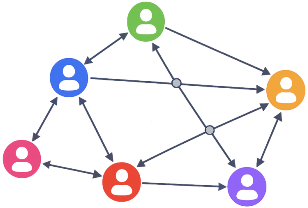

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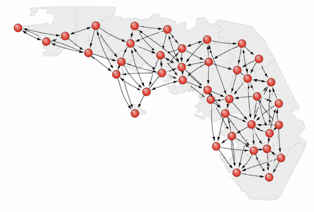

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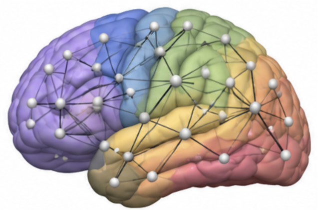

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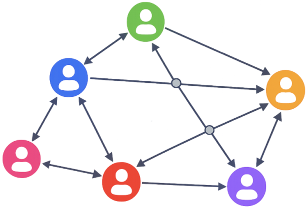

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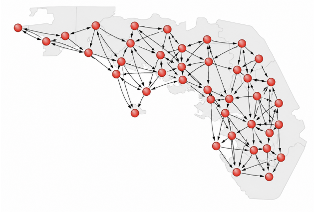

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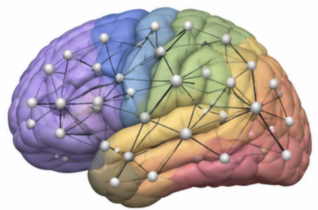

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- Applications in various cases (e.g., social networks, sensor networks, brain connectomics).
- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$: directed graph with N nodes and adjacency matrix $\mathbf{A}$.
- Eigendecomposition $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$, where $\mathbf{U}$ is the eigenvector matrix and $\boldsymbol{\Lambda}$ the diagonal eigenvalue matrix.
- Graph Fourier Transform of a graph signal $\mathbf{x}$: $\hat{\mathbf{x}} = \mathbf{U}^{-1}\mathbf{x}$.

Motivations: Noise Model

- Usual formulation:

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\epsilon},$$

  where the clean signal $\mathbf{x}$ is a “domain related signal”, $\boldsymbol{\epsilon}$ is “domain related noise”. Denoised by various filters (e.g. Wiener, Tikhonov, Kalman etc...) that leverage the domain structure.

Motivations: Noise Model

- Observation model:

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\epsilon_s} + \boldsymbol{\epsilon_d},$$

  where the clean signal $\mathbf{x}$ is a "graph related signal", $\boldsymbol{\epsilon_s}$ is "graph related noise", and $\boldsymbol{\epsilon_d}$ is measurement noise.

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- Denoising problem $\implies$ given $\mathbf{y}$, find graph filter to minimize the expected quadratic error:

$$\mathbf{G}_{\text{opt}} = \underset{\mathbf{G} \in \mathcal{G}}{\arg\min}\, \mathbb{E}[||\mathbf{G}\mathbf{y} - \mathbf{x}||^2],$$

  where $\mathcal{G}$ is the set of graph filters.

w.l.g mean vector is $\mu_{\mathbf{x}} = \mathbb{E}[\mathbf{x}] = \mathbf{0}$ and thus covariance matrix is $\mathbf{H}_{\mathbf{x}}[i,j] = \mathbb{E}\left[\mathbf{x}[i]\mathbf{x}[j]\right]$.

Definition (Directed Graph Wide Sense Stationarity)

A random signal $\mathbf{x}$ is said to be *directed graph wide sense stationary* (DGWSS) if the following conditions hold:

$$\mathbf{H}_{\mathbf{x}} = \mathbf{U}\hat{\mathbf{H}}_{\mathbf{x}}\mathbf{U}^H,$$

where $\hat{\mathbf{H}}_{\mathbf{x}}$ is a diagonal matrix. Its diagonal entries $\boldsymbol{\gamma}_{\mathbf{x}}[k] = \hat{\mathbf{H}}_{\mathbf{x}}[k,k]$ define the graph power spectral density (PSD). Equivalently, $\hat{\mathbf{H}}_{\mathbf{x}} = \mathbf{U}^H\mathbf{H}_{\mathbf{x}}\mathbf{U} = \mathbb{E}\{\hat{\mathbf{x}}\hat{\mathbf{x}}^H\}$ being diagonal, shows that the spectral components of $\mathbf{x}$ are decorrelated ([Iraji et al., 2025, Chan et al., 2026]).

This provides an alternative definition for WSS signals in time-domain for which the graph is a cycle.

Graph White Noise

Definition (White Noise)

A random signal $\mathbf{w}$ is said to be *white noise* (WN) if it is WSS and its power spectrum is 1 on all frequencies.

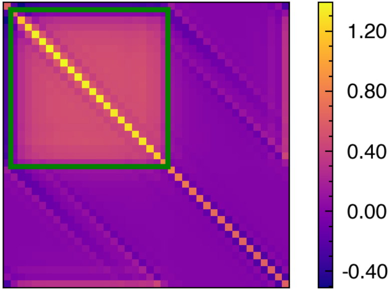
Definition (Directed Graph White Noises)

A random graph signal $\mathbf{w}$ is said to be *directed graph white noise* if it is DGWSS and its PSD is $\mathbf{1}$, i.e., $\hat{\mathbf{H}}_{\mathbf{w}} = \mathbf{I}_N$.

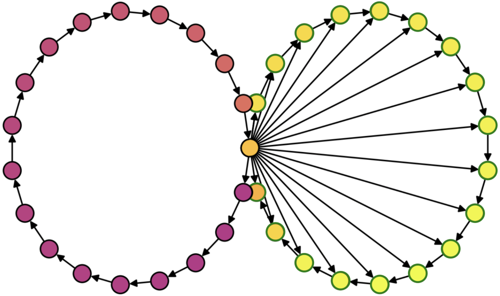
Note that for any DGWSS $\mathbf{x}$ with PSD $\hat{\mathbf{H}}_{\mathbf{x}}$, we can write

$$\mathbf{x} = \mathbf{U}\hat{\mathbf{H}}_{\mathbf{x}}^{1/2}\mathbf{U}^{-1}\mathbf{w} = \mathbf{H}\mathbf{w},$$

where $\mathbf{H}$ is a graph filter and $\mathbf{w}$ is a directed graph WN.



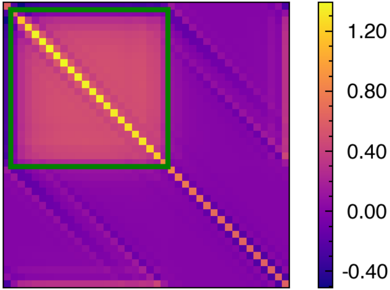
(a) Covariance structure



(b) Variance on the graph

- High in/out degree nodes have higher variance.
- Higher covariance between child nodes of the same parent node.

Graph White Noise



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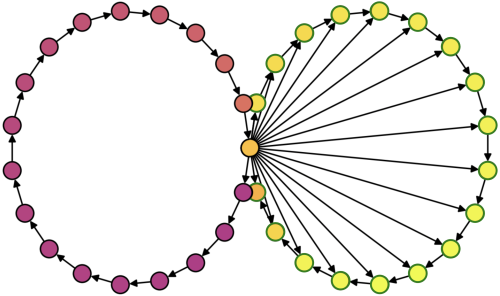

(b) Variance on the graph

- Second moments of graph white noise are structure dependant [Chan et al., 2026].
- Non WN also carry graph structure, as it is the result of graph filtering WN.

Optimal Graph Filter

- Observation model:

$$\mathbf{y} = \mathbf{x} + \boldsymbol{\epsilon_s} + \boldsymbol{\epsilon_d},$$

  where the clean signal $\mathbf{x}$ is a "**graph related signal**", $\boldsymbol{\epsilon_s}$ is "**graph related noise**", and $\boldsymbol{\epsilon_d}$ is **measurement noise**.

- Denoising problem $\implies$ given $\mathbf{y}$, find graph filter to minimize the expected quadratic error:

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  where the clean signal $\mathbf{x}$ is a **DGWSS**, $\boldsymbol{\epsilon_s}$ is **directed graph WN** (structured), and $\boldsymbol{\epsilon_d}$ is **WN** (decorrelated).

- Denoising problem $\implies$ given $\mathbf{y}$, find graph filter to minimize the expected quadratic error:

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Optimal Graph Filter

Proposition (Wiener Filter)

The graph Wiener spectral filter $\hat{\mathbf{G}}_{opt}$ *is determined by*

$$diag\left(\hat{\mathbf{G}}_{opt}\right) = \left(\hat{\mathbf{H}}_{\mathbf{x}} + \hat{\mathbf{H}}_{\boldsymbol{\epsilon}_s} + \mathbf{M}\right)^{-1} \boldsymbol{\gamma}_{\mathbf{x}},$$

where $\mathbf{M} = \left((\mathbf{U}^{-1}\mathbf{H}_{\boldsymbol{\epsilon}_d}\mathbf{U}^{-H}) \circ (\mathbf{U}^H\mathbf{U})^T\right)$, *is an optimal graph filter for expected quadratic error w.r.t.* $\mathbf{y}$ *and* $\mathbf{x}$.

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$\hat{\mathbf{G}}_{opt}$ *is complex-valued and all real signal filtered by* $\hat{\mathbf{G}}_{opt}$ *remain real.*

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When no decorrelated noise is present, it leads to the classical graph Wiener filter ([Iraji et al., 2025, Marques et al., 2017, Perraudin and Vandergheynst, 2017]):

$$\hat{g}[k] = \frac{\gamma_{\mathbf{x}}[k]}{\gamma_{\mathbf{x}}[k] + \gamma_{\boldsymbol{\epsilon}}[k]}.$$

Preliminary Experiments

Starting from the observation model, we define the random graph signals

$$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, a^2\,\mathbf{U}\hat{\mathbf{H}}_{\mathbf{x}}\mathbf{U}^H),$$
$$\boldsymbol{\epsilon}_s \sim \mathcal{N}(\mathbf{0}, b^2\,\mathbf{U}\mathbf{U}^H),$$
$$\boldsymbol{\epsilon}_d \sim \mathcal{N}(\mathbf{0}, c^2\,\mathbf{I}_N),$$

with $a, b, c \in \mathbb{R}$ and $\boldsymbol{\gamma}_{\mathbf{x}}[k] = e^{-k}$.

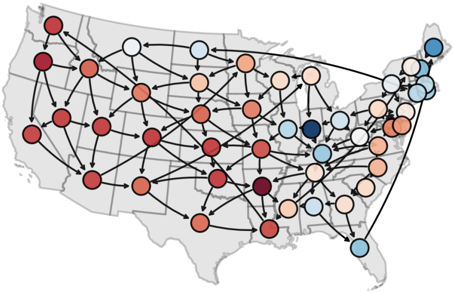

Figure: Graph visualization and sample graph signal.

Preliminary Experiments

For a fixed SNR, we compare the denoising performances of undirected (U), directed (D), optimal Wiener (O) and undirected (U), directed (D) Tikhonov filters.

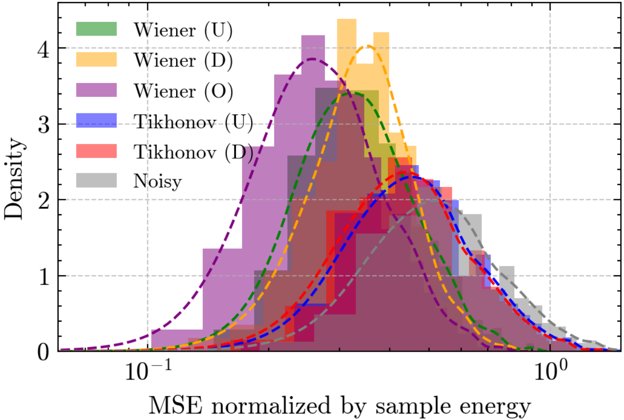

Figure: Comparison of denoising results with $a = 5, b = 1.5, c = 1$.

We also investigate the influence of noises level on the denoising performance.

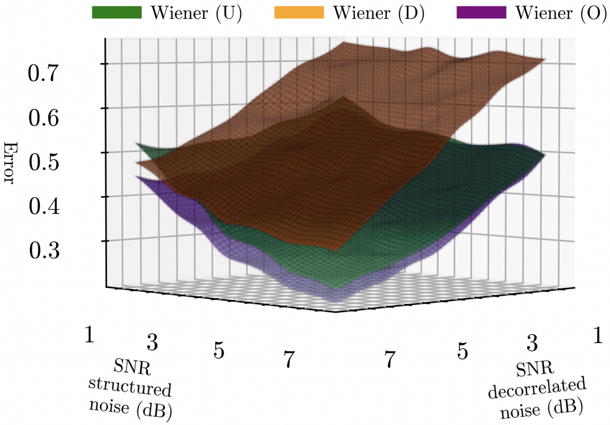

Figure: Influence of SNR on the denoising performance.

Discussions

- The optimal Wiener filter is robust against mixed structured and decorrelated noise.

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- It also handles noise types that are not necessarily decorrelated, provided the covariance structure can be estimated (e.g. symmetrized graph's WN).

Discussions

- The optimal Wiener filter is robust against mixed structured and decorrelated noise.
- Denoising is not limited to WN — it extends naturally to colored noise.
- It also handles noise types that are not necessarily decorrelated, provided the covariance structure can be estimated (e.g. symmetrized graph's WN).
- Next steps include testing on real-world datasets, and testing the different cases of colored noise, and non-decorrelated noise.

References

List of references:

Chan, C. H. M., Cionca, A., and Van De Ville, D. (2026).
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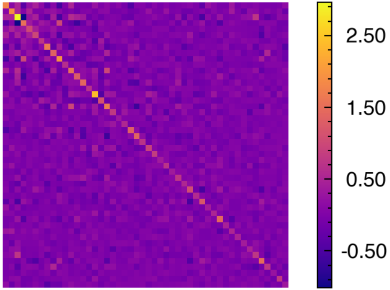
Thank you for your attention!

Appendix: Structured WN



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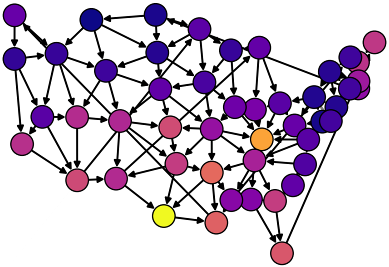

(b) Variance on the graph

- Higher variance for southern nodes as these nodes have higher in/out degree.

Appendix: Deconvolution

While we mainly interest ourselves in the denoising, it is straightforward to extend the formulation of the directed graph Wiener filter to include deconvolution. The corresponding observation model is

$$\mathbf{y} = h(\mathbf{A})\mathbf{x} + \boldsymbol{\epsilon}_s + \boldsymbol{\epsilon}_d$$

where $h(\mathbf{A}) \in \mathbb{R}^{N \times N}$ is a nodal filter.

Proposition

The graph Wiener filter $\hat{\mathbf{G}}_{opt}$ is determined by

$$\mathit{diag}\left(\hat{\mathbf{G}}_{opt}\right) = \left(|h(\boldsymbol{\Lambda})|^2 \hat{\mathbf{H}}_{\mathbf{x}} + \hat{\mathbf{H}}_{\boldsymbol{\epsilon}_s} + \mathbf{M}\right)^{-1} |h(\boldsymbol{\Lambda})| \boldsymbol{\gamma}_{\mathbf{x}}$$

is an optimal graph filter for expected quadratic error w.r.t to $\mathbf{x}$.

Proposition

The Tikhonov filter $\mathbf{G}_{\textit{tikhonov}}$ *is determined by*

$$\mathbf{G}_{\textit{tikhonov}} = (\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\epsilon}_s+\boldsymbol{\epsilon}_d} + \mathbf{L}^H\mathbf{L})^{-1}\boldsymbol{\Sigma}^{-1}_{\boldsymbol{\epsilon}_s+\boldsymbol{\epsilon}_d},$$

with $\mathbf{L} = \mathbf{I} - \mathbf{A}$.

On the assumption of prior and noise distributions:

$$\begin{aligned} \mathbf{x} &\sim \mathcal{N}(\mathbf{0}, \mathbf{L}^{\dagger}(\mathbf{L}^{\dagger})^H), \\ \boldsymbol{\epsilon}_s + \boldsymbol{\epsilon}_d &\sim \mathcal{N}(\mathbf{0}, \Sigma_{\epsilon_s+\boldsymbol{\epsilon}_d}). \end{aligned}$$