Sampling in the Graph Signal Processing Companion Model

GSP Workshop 2026

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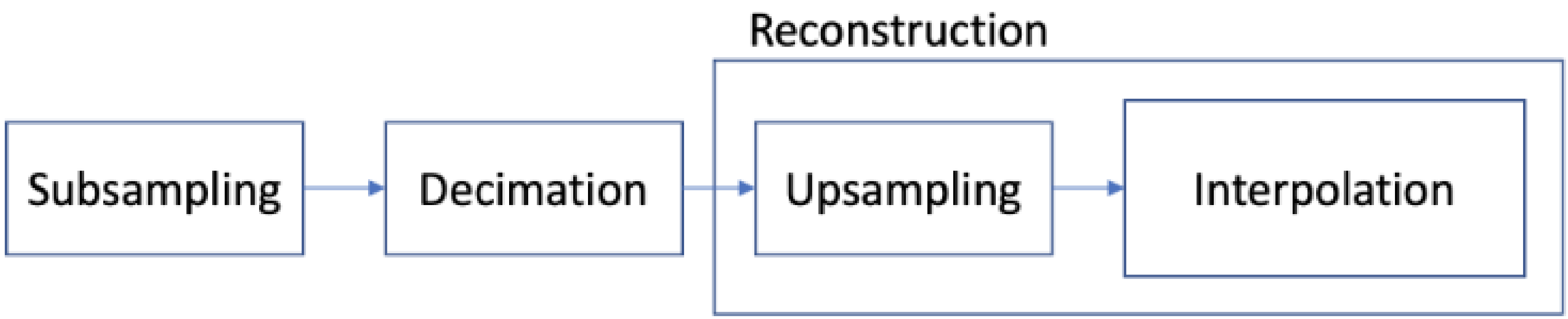
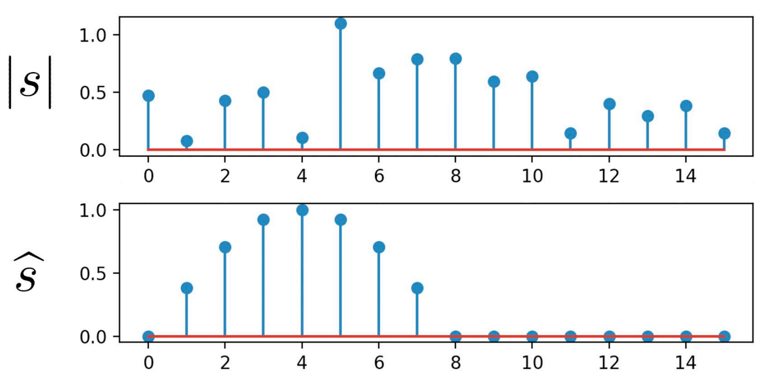
Outline

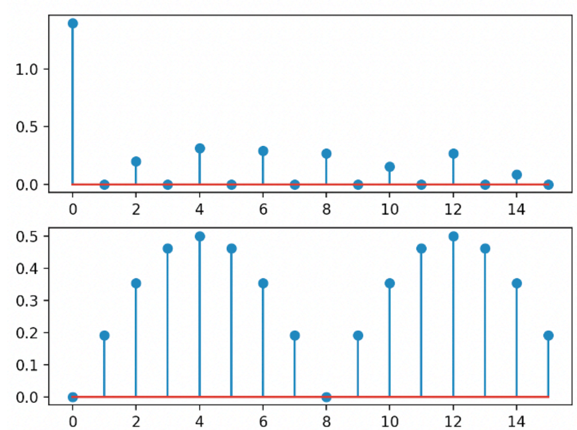
I. GSP Sampling in the Vertex / Spectral Domain
II. GSP Companion Model
III. Sampling in the Companion Model

DSP Sampling

In DSP, “(Uniform) sampling of a bandlimited signal leads to (perfect) replication in the spectral domain.”

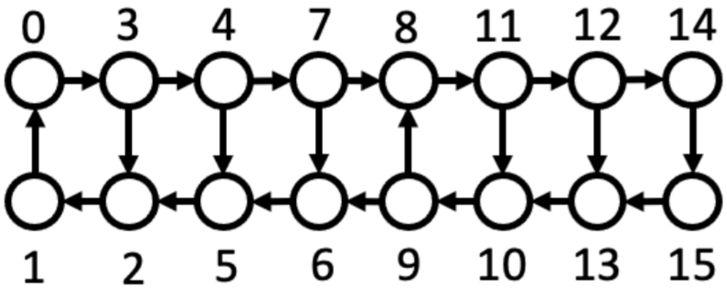


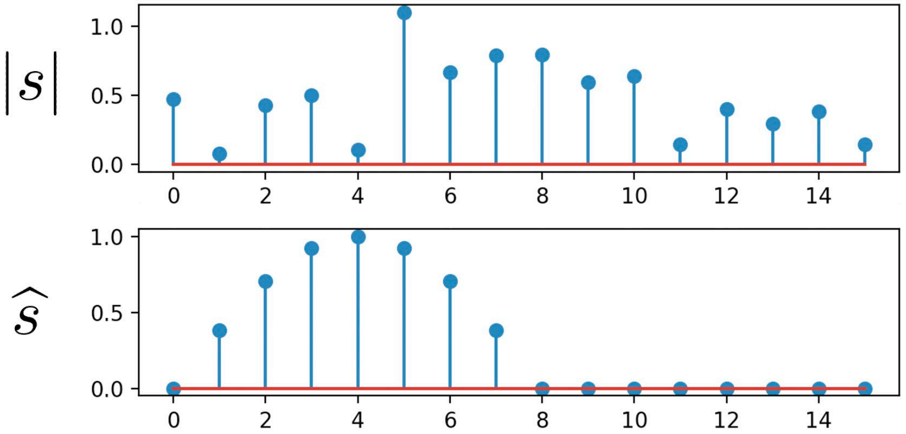




GSP Sampling

How can we sample this?



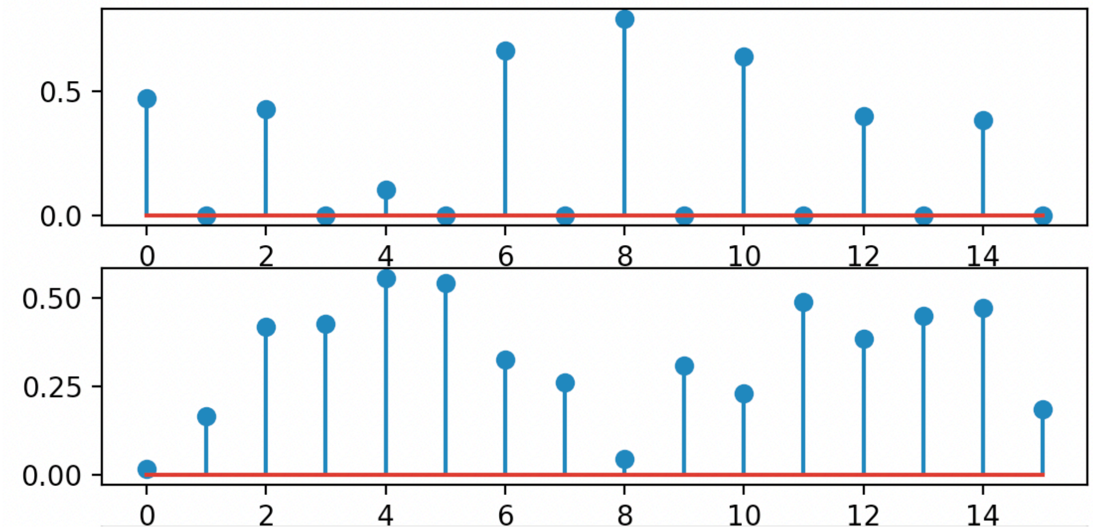


GSP Sampling

Approach 1: Sampling in the Vertex Domain

Vertex

Spectral



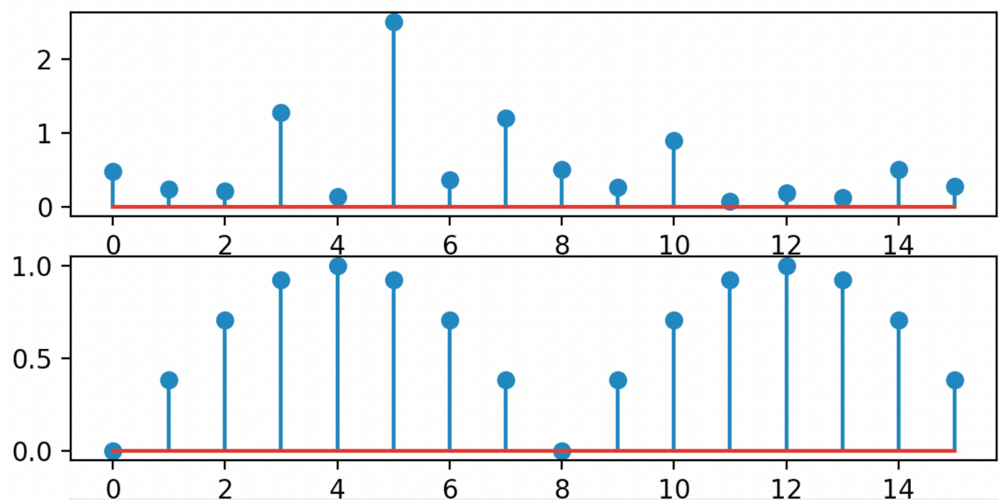
Sample values in s

Distorted Replicas

Approach 2: Perfect Replication in the Spectral Domain

Vertex

Spectral



Not Sampled Values

Perfect Replicas

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GSP Sampling - Problems

- Neither GSP subsampling method produces sampling in the vertex domain and perfect replication in the spectral domain.

- No Fixed Ordering makes it impossible to “uniformly” sample graph signals.

Research Questions

Under what conditions, does GSP Sampling become DSP sampling?

How can we convert the 4 steps of sampling from DSP to GSP with perfect replication?

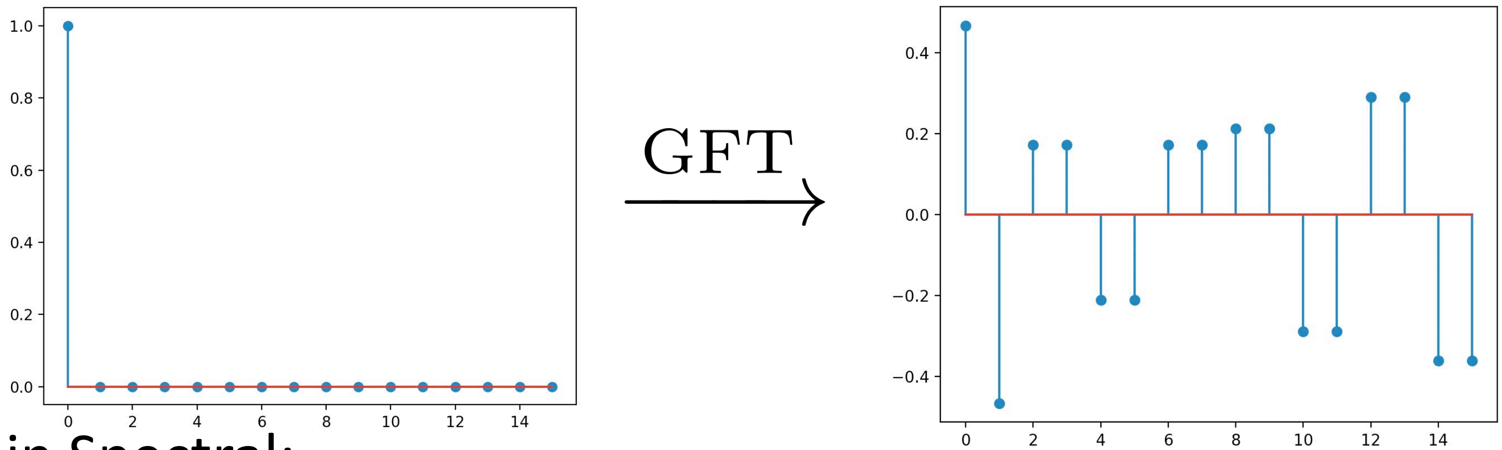
Answer: Use the GSP Companion Model

The GSP Companion Model

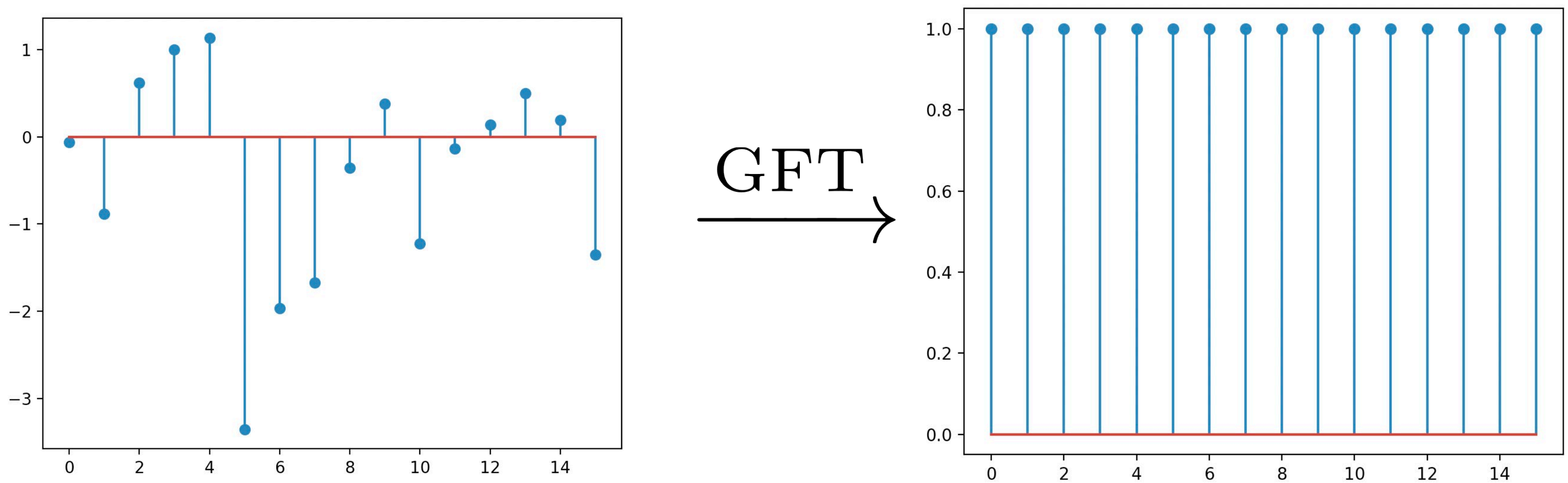
How do we design new GSP concepts using DSP intuition?

DSP to GSP: Impulse Functions

Impulsive in Vertex:



Flat in Spectral:



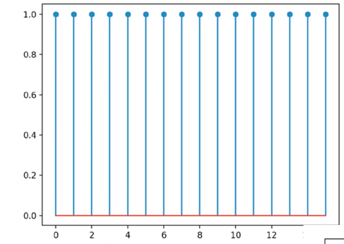
Impulse Functions

Impulse Function: Flat in Spectral

$$\widehat{\delta}_0 = [1, 1, \ldots, 1]^T$$

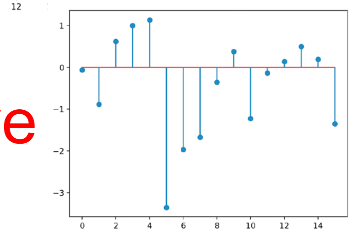
$$\delta_0 = \mathrm{GFT}^{-1}[1, 1, \ldots, 1]^T$$

Flat



$\widehat{\delta}_0$

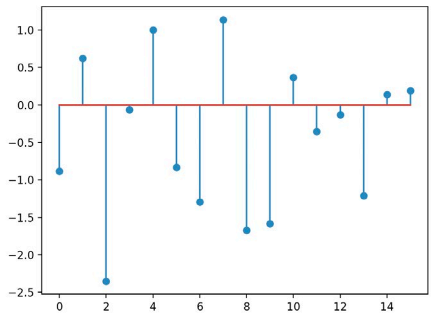
NOT Impulsive



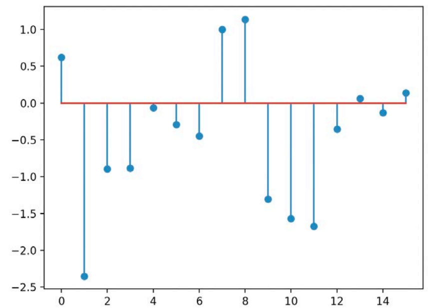
δ_0

$$\delta_i = A^i \delta_0, i = 0, 1, \ldots, N-1$$

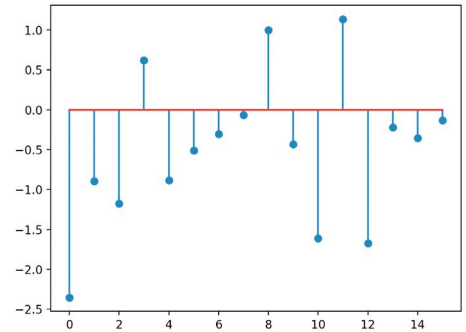
Shifted Impulses



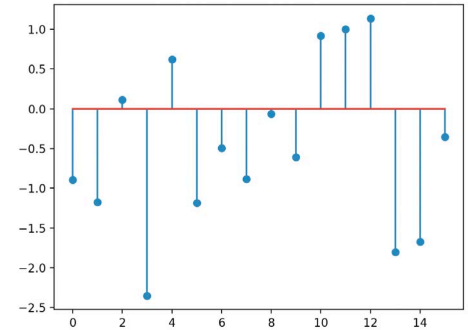
$\delta_1 = A\delta_0$



$\delta_2 = A^2\delta_0$



$\delta_3 = A^3\delta_0$



$\delta_4 = A^4\delta_0$

. . .

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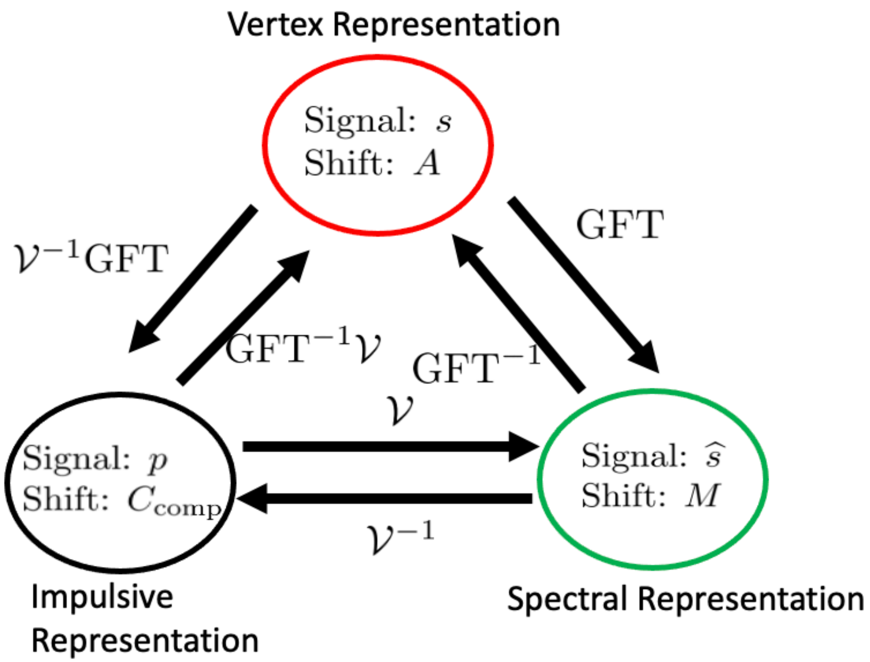
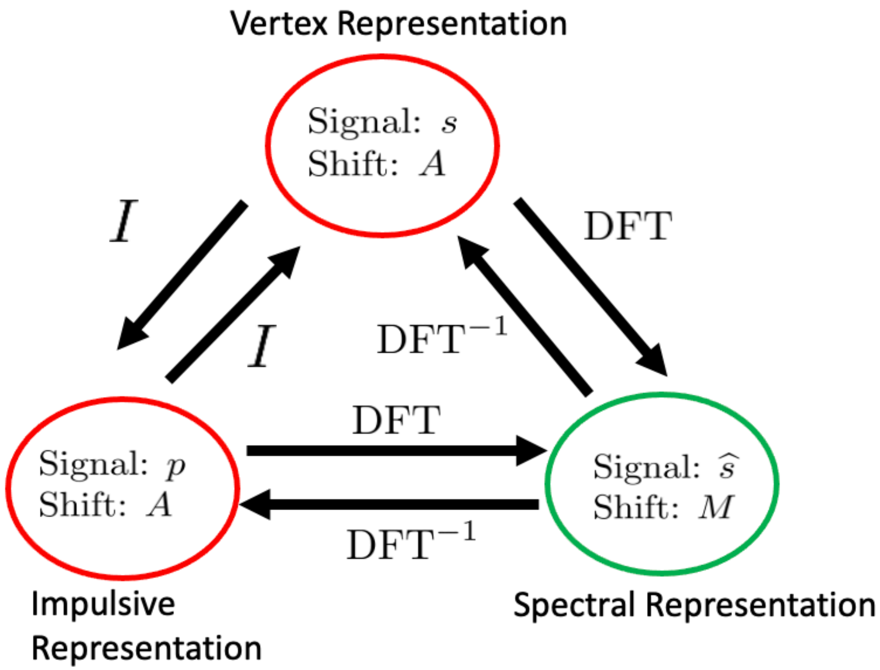
We use these shifted impulses as our basis.

Three Signal Representations





$$\mathcal{V} = \left[\lambda^0 \cdots \lambda^{N-1}\right] = \begin{bmatrix} 1 & \lambda_0 & \lambda_0^2 & \cdots & \lambda_0^{N-1} \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{N-1} & \lambda_{N-1}^2 & \cdots & \lambda_{N-1}^{N-1} \end{bmatrix}.$$

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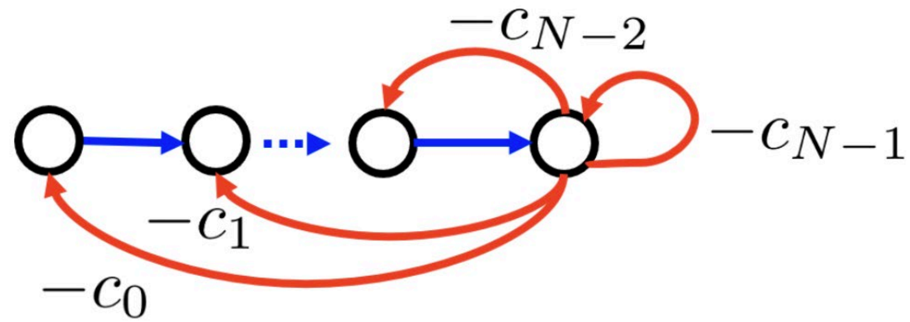
GSP = DSP + B.C.

$$C_{\text{comp}} = \begin{bmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{N-1} \end{bmatrix}$$



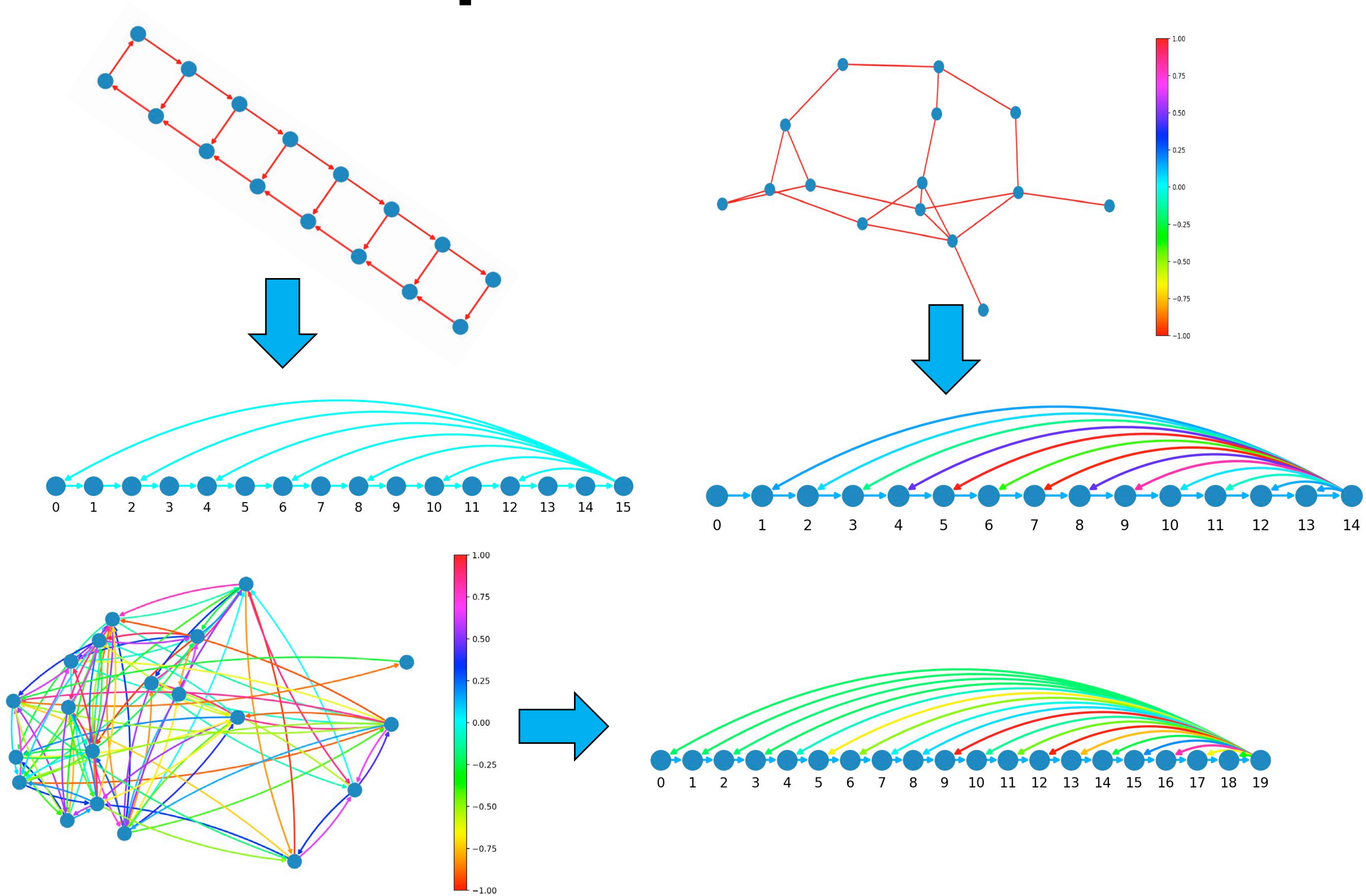
Path + Boundary Conditions (B.C)

$$\Delta_A(x) = |xI - A| = x^N + c_{N-1}x^{N-1} + \dots + c_0$$

The GSP Companion Model (z-transform) mimics DSP to the largest extent possible.

J. Shi and J. M. F. Moura, “GSP = DSP + boundary conditions: The GSP companion model,” in arXiv preprint: 2203.13791, March 2023. Submitted for publication.

Companion Model



Regardless of A, consistent format for $C_{comp.}$

Design Using the Companion Model

The Companion Model connects DSP and GSP concepts, revealing new insights about both DSP and GSP.

Any graph (with unique eigenvalues) can become a companion graph (path + b.c.).

When designing new GSP concepts:

1. Design in the Companion Model to replicate DSP to the largest extent possible (considering boundary conditions).
2. Extend the design into existing GSP vertex and spectral domains.

GSP Companion Model: Sampling

Sampling:

Can Uniformly Sample in Companion Model because of fixed ordering: $s = p_0\delta_0 + p_1\delta_1 + \ldots + p_{N-1}\delta_{N-1}$

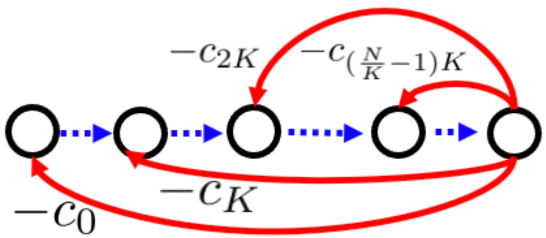
The Step 1-Step 2 Companion Model Method (S^2CM^2) [2] allows for the design of new GSP concepts in the companion model.

GSP Companion Model: Sampling

Sampling Set: Every K values, K | N

Assumption: Characteristic Polynomial is

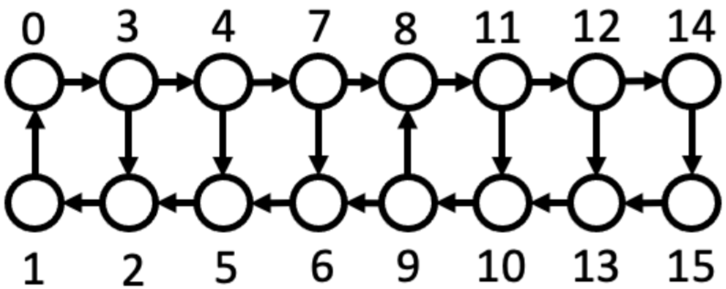
$$\Delta_A(x) = x^N + c_{((\frac{N}{K})-1)K} x^{((\frac{N}{K})-1)K} + \ldots + c_K x^K + c_0$$

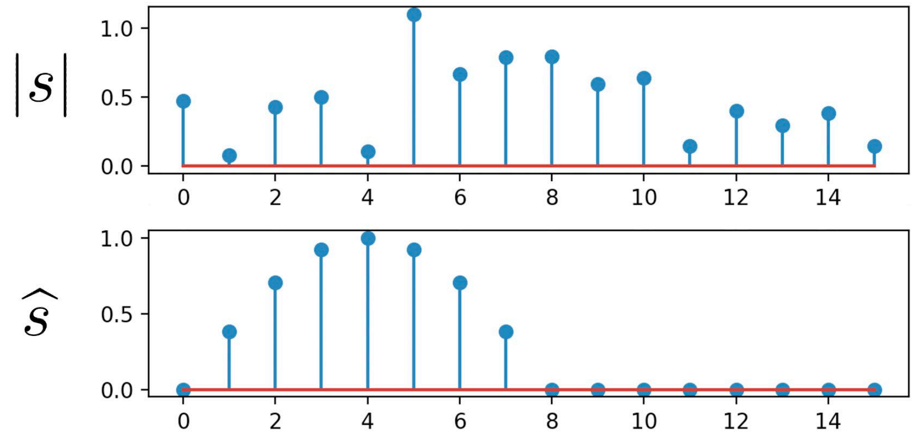


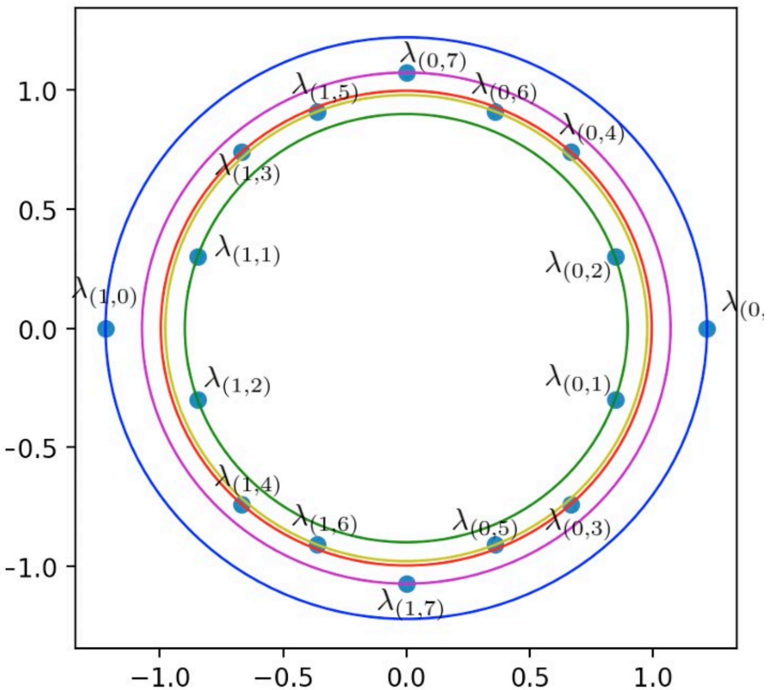
Equivalent to:

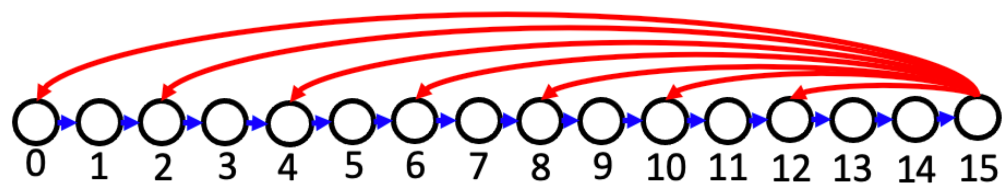
Sets of K Eigenvalues are evenly spaced around circles in $\mathbb{C}$

GSP Companion Model: Sampling









Characteristic Polynomial is

$$x^{16} - x^{12} - x^{10} - x^8 - x^6 - x^4 - x^2 - 1 = 0$$

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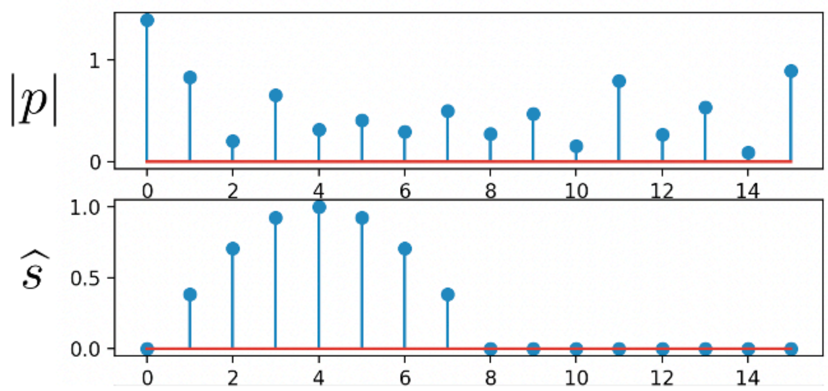
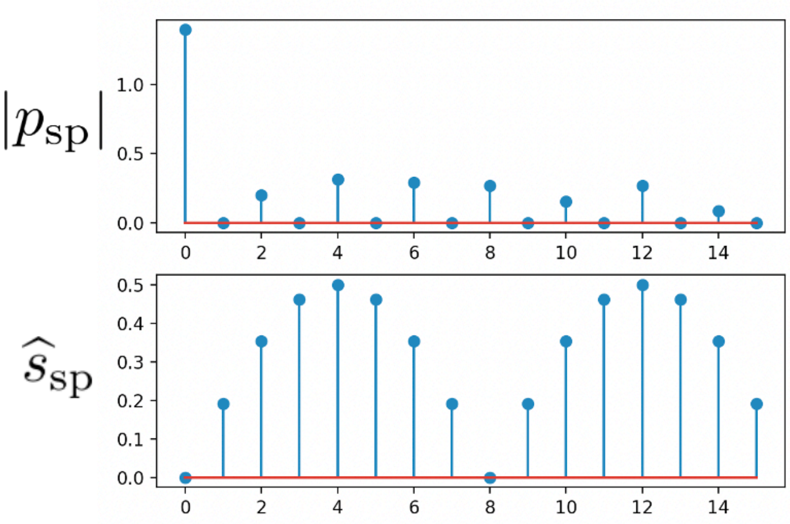
GSP Companion Model: Subsampling

Theorem 1: (Perfect replicas of a bandlimited signal) If $\widehat{s}$ is bandlimited with bandlimit $\frac{N}{K}$ then sampling *p* produces perfect replicas of the band in the spectral domain.





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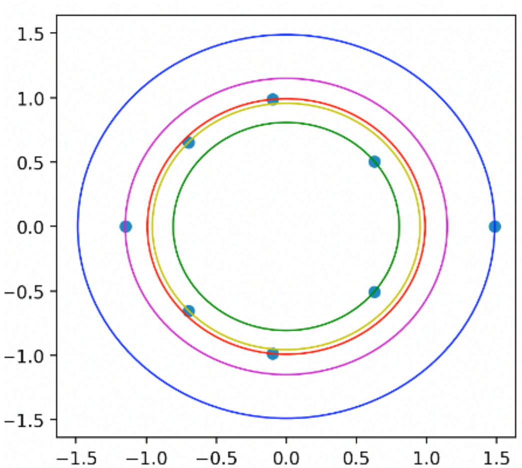
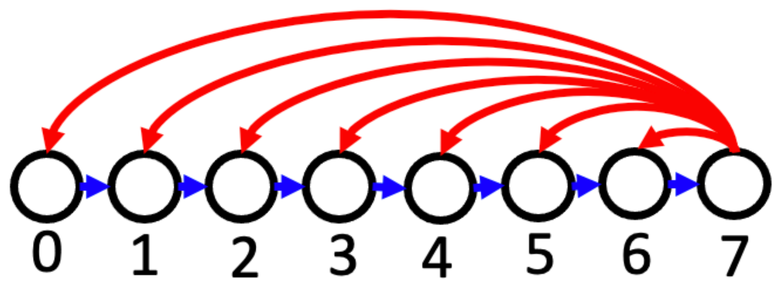
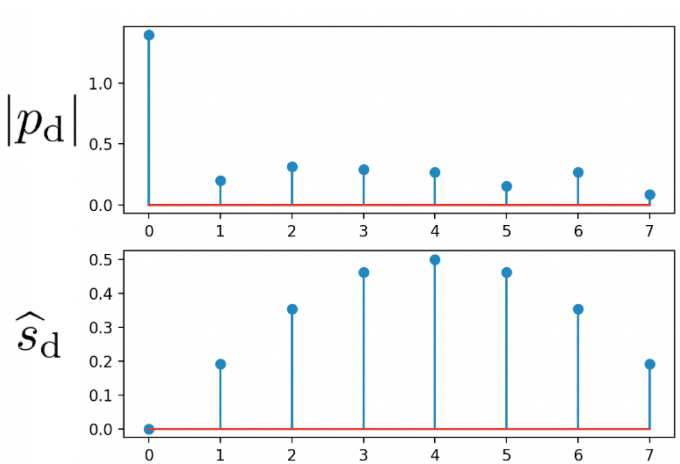
GSP Companion Model: Decimation

Raise Eigenvalues to the Kth power. Remove zeros.







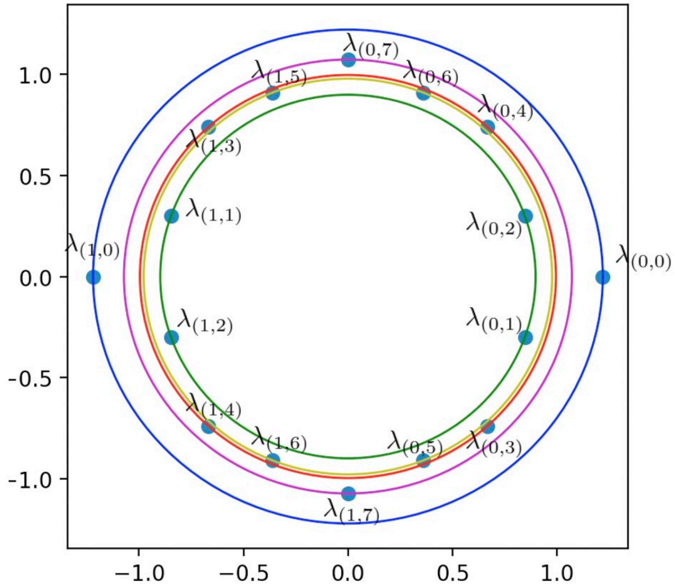
$$x^{16} - x^{12} - x^{10} - x^{8} - x^{6} - x^{4} - x^{2} - 1 = 0$$

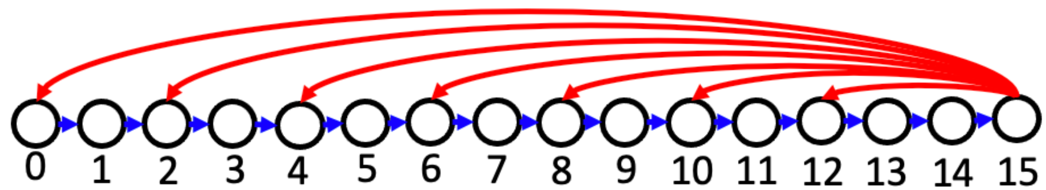
$$x^{8} - x^{6} - x^{5} - x^{4} - x^{3} - x^{2} - x - 1 = 0$$

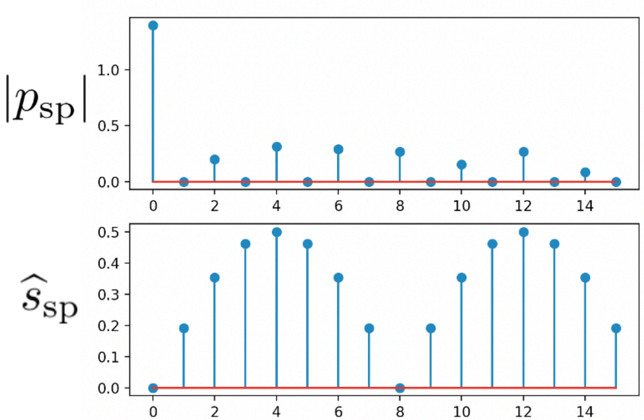
GSP Companion Model: Upsampling

Take the Kth roots of the Eigenvalues.

Reintroduce zeros to p.





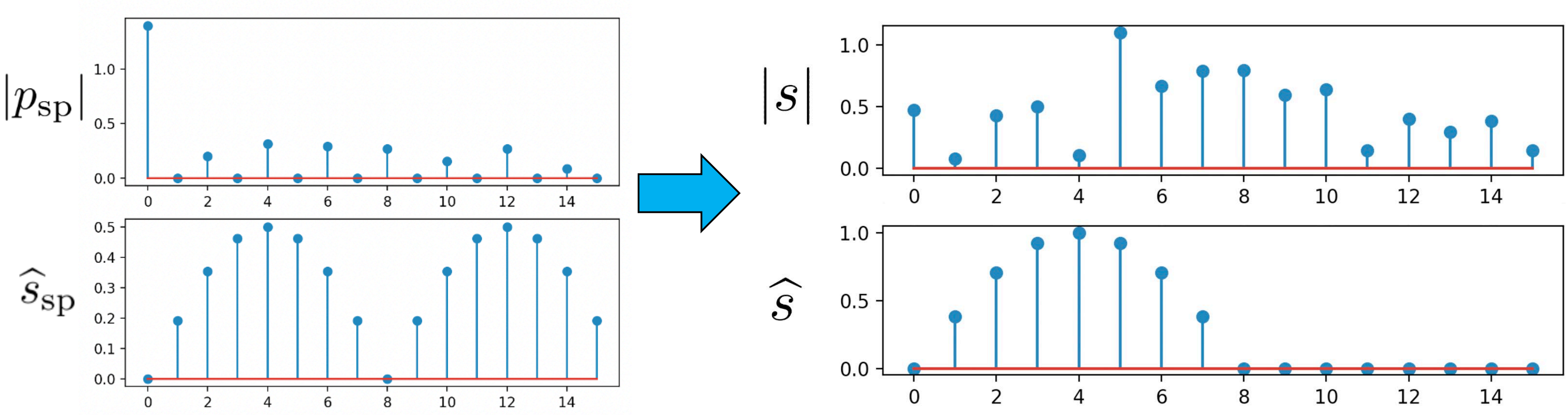


$$x^{16} - x^{12} - x^{10} - x^{8} - x^{6} - x^{4} - x^{2} - 1 = 0$$

$$x^{8} - x^{6} - x^{5} - x^{4} - x^{3} - x^{2} - x - 1 = 0$$

GSP Companion Model: Interpolation

Apply a low pass filter in frequency to obtain the original signal



Conclusion

- When sampling in the GSP companion model, we achieve perfect replication in the spectral domain, which is not observed in the traditional GSP model beyond a few special cases.
- By converting to the companion model, we can design GSP concepts with DSP intuition.
- Using the companion model, if the characteristic polynomial is

$$\Delta_A(x) = x^N + c_{((\frac{N}{K})-1)K} x^{((\frac{N}{K})-1)K} + \ldots + c_K x^K + c_0$$

then, the perfect replication allows for the four sampling steps in the companion model to be similar to the four DSP sampling steps.