Möbius Model for Graph Signal Processing on Weighted DAGs

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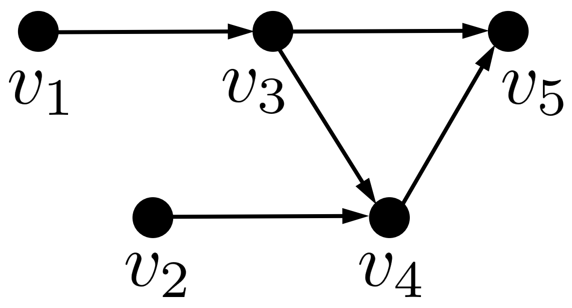
Directed acyclic graphs (DAGs)

All eigenvalues of A are zero.

Fourier basis is undefined.

Graph signal processing is ill-defined for DAGs.



$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

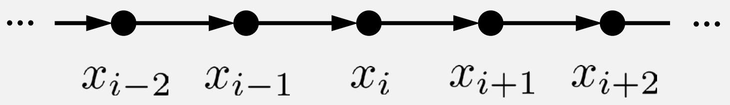
Goal: define a GSP model for DAGs

Our contribution

- Novel orthogonal Fourier transform for weighted DAGs
- Generalizes the prior Möbius model for unweighted DAGs
- Application: denoising of DAG signals

Background: Möbius model for unweighted DAGs

Discrete-time (is a DAG)



Defines an order

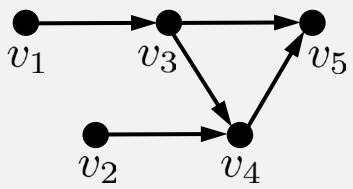
Total variation: $\mathrm{TV}(\mathbf{x}) = \sum_i |x_i - x_{i-1}| = \|\Delta \mathbf{x}\|_1$

Difference operator: $(\Delta \mathbf{x})_i = x_i - x_{i-1}$

sums over all predecessors

Integration operator: $(\Sigma \mathbf{x})_i = (\Delta^{-1}\mathbf{x})_i = \sum_{j \leq i} x_j$

Directed acyclic graphs



Defines a partial order

$v_1 \preceq v_3, v_4, v_5$

$v_2 \preceq v_4, v_5$

$v_3 \preceq v_4, v_5$

$v_4 \preceq v_5$

$\mathrm{TV}(\mathbf{x}) = \|M\mathbf{x}\|_p^p$ — Möbius TV

$\Delta = Z^{-1} = M$ — Möbius difference operator

$(Z\mathbf{x})_i = \sum_{v_j \preceq v_i} x_j$ — Zeta integration operator

Background: Möbius model for unweighted DAGs

Möbius Fourier basis

Eigenbasis of $M^\top M$ (or $ZZ^\top$)

Orthogonal

Iteratively minimizes Möbius TV

integration operator

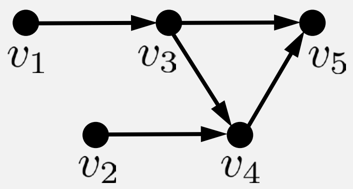
↓

difference operator

↓

Fourier basis

Directed acyclic graphs



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Background: Möbius model for unweighted DAGs

Möbius Fourier basis

Eigenbasis of $M^\top M$ (or $ZZ^\top$)

Orthogonal

Iteratively minimizes Möbius TV

integration operator

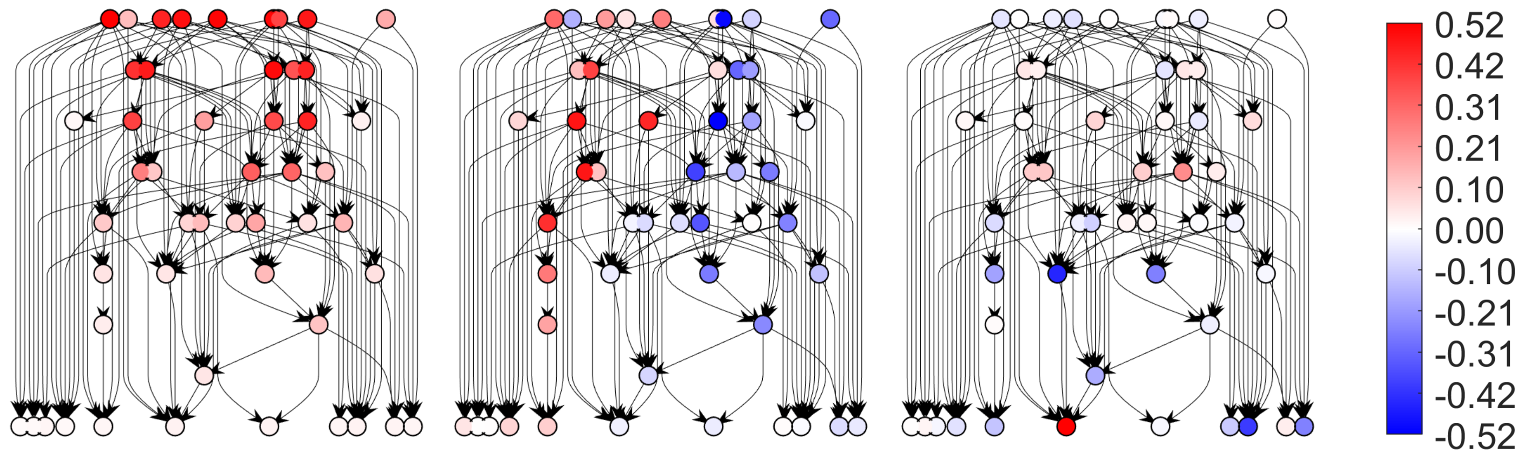
⬇

difference operator

⬇

Fourier basis

Example

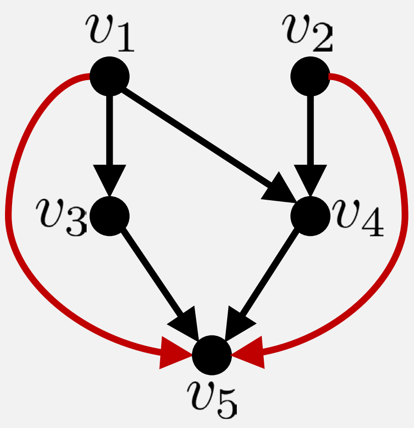


1st vector　　2nd vector　　Last vector

Basis vectors capture a special notion of smoothness.

Generalization to weighted DAGs

Transitive closure



$$\overline{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

transitive-reflexive closure ≡ integration operator

Key idea: generalize transitive-reflexive closure to weighted DAGs

Boolean semiring

Adjacency matrix of an unweighted DAG is a matrix over the *Boolean semiring*.

$$a \oplus b = a \text{ OR } b$$
$$a \odot b = a \text{ AND } b$$
$$a, b \in \{0, 1\}$$

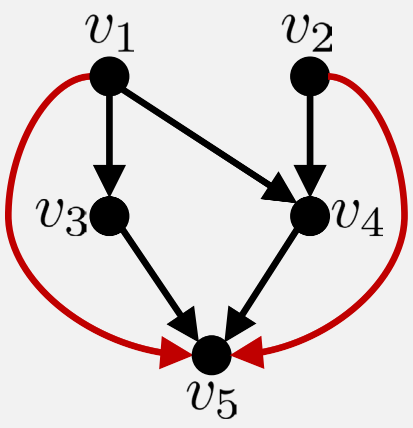
Transitive closure computed with the Floyd-Warshall algorithm:

Input: Adjacency matrix A of a DAG on n vertices
Output: Transitive closure $\overline{A}$

$A^{(0)} \leftarrow A$
for $k = 1$ to n **do**
  for $i = 1$ to n **do**
    for $j = 1$ to n **do**
      $A^{(k)}_{i,j} \leftarrow A^{(k-1)}_{i,j} \oplus (A^{(k-1)}_{i,k} \odot A^{(k-1)}_{k,j})$
return $\overline{A} \leftarrow A^{(n)}$

Generalization to weighted DAGs

Transitive closure



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Key idea: generalize transitive-reflexive closure to weighted DAGs

Generalize the Boolean semiring

Boolean semiring

Adjacency matrix of an unweighted DAG is a matrix over the *Boolean semiring*.

$$a \oplus b = a \text{ OR } b \qquad a, b \in \{0, 1\}$$
$$a \odot b = a \text{ AND } b$$

Transitive closure computed with the Floyd-Warshall algorithm:

Input: Adjacency matrix A of a DAG on n vertices
Output: Transitive closure $\overline{A}$

```
A^(0) ← A
for k = 1 to n do
    for i = 1 to n do
        for j = 1 to n do
            A_{i,j}^(k) ← A_{i,j}^(k−1) ⊕ (A_{i,k}^(k−1) ⊙ A_{k,j}^(k−1))
return Ā ← A^(n)
```

Matrices over semirings

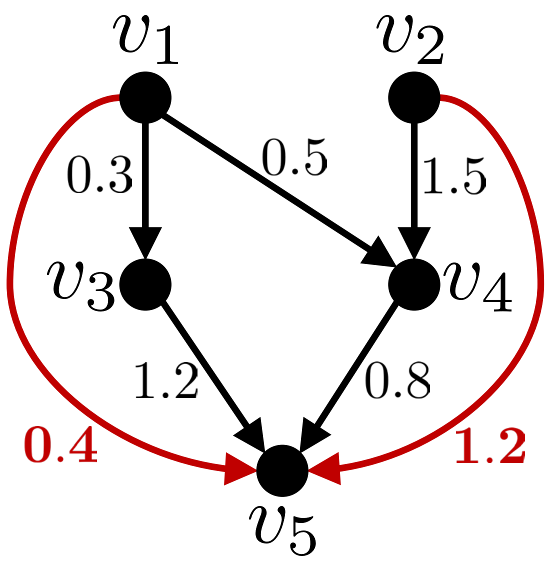
Max-times semiring generalizes the Boolean semiring because

$$\max(a,b) = a \text{ OR } b, \quad a \cdot b = a \text{ AND } b \quad \text{for } a,b \in \{0,1\}$$

Semiring	**Domain S**	$a \oplus b$	$a \odot b$	0_S	1_S
Boolean	$\{0,1\}$	a OR b	a AND b	0	1
Max-times	$\mathbb{R}_{\geq 0}$	$\max(a,b)$	$a \cdot b$	0	1
Max-min	$\mathbb{R} \cup \{\pm\infty\}$	$\max(a,b)$	$\min(a,b)$	$-\infty$	∞
Min-plus	$\mathbb{R} \cup \{\pm\infty\}$	$\min(a,b)$	$a+b$	∞	0
Max-plus	$\mathbb{R} \cup \{\pm\infty\}$	$\max(a,b)$	$a+b$	$-\infty$	0
Arithmetic	$\mathbb{R} \cup \{\infty\}$	$a+b$	$a \cdot b$	0	1

Möbius model for weighted DAGs

Max-times semiring generalizes the Boolean semiring because
$\max(a,b) = a \text{ OR } b, \quad a \cdot b = a \text{ AND } b$ for $a, b \in \{0,1\}$



$$\overline{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 \\ 0.5 & 1.5 & 0 & 0 & 0 \\ 0.4 & 1.2 & 1.2 & 0.8 & 0 \end{pmatrix}$$

$$0.4 = \max(0.3 \cdot 1.2,\ 0.5 \cdot 0.8)$$

$$Z = \overline{A} + I$$

integration operator

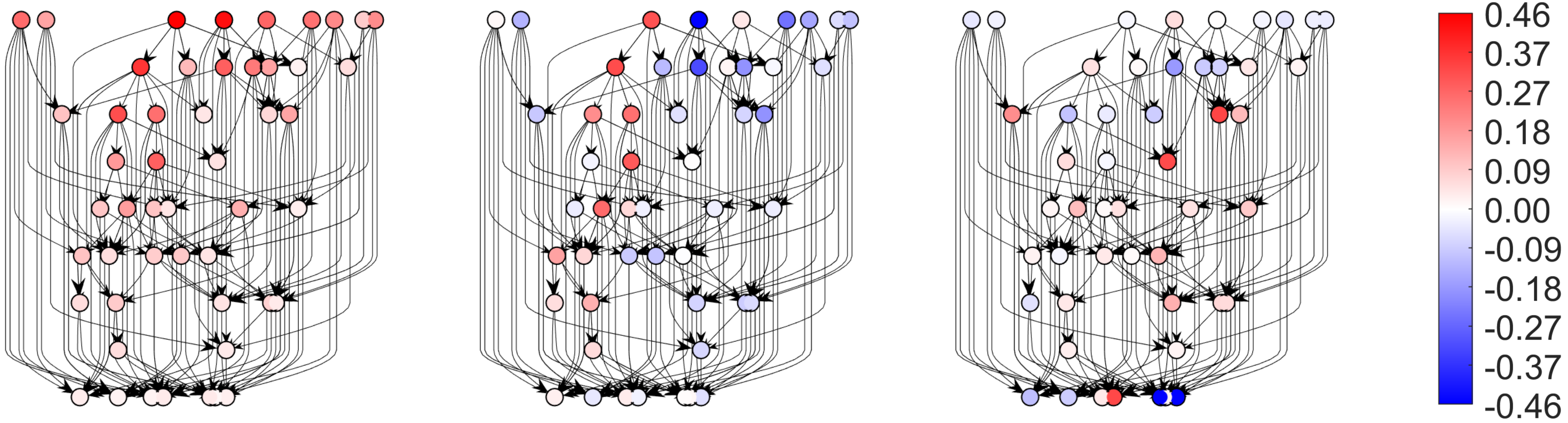
$$M = Z^{-1}$$

difference operator

transitive-reflexive closure ≡ integration operator

integration operator ➞ difference operator ➞ Fourier basis (eigenbasis of $M^\top M$)

Example



1st vector 2nd vector Last vector

Möbius model variants

Sourceless Möbius operator: $\widehat{M} = \mathrm{diag}(\mathbf{1} - \mathbf{1}_{\mathcal{S}}) \cdot M$

Balanced Möbius operator: $\widetilde{M} = M \cdot \left(I - \frac{\mathbf{1}\mathbf{1}_{\mathcal{S}}^{\top}}{|\mathcal{S}|} \right)$

Application: Signal Denoising

Noise model: $\mathbf{y} = \mathbf{x} + \mathbf{e}$

- $\mathbf{y}$: noisy
- $\mathbf{x}$: original
- $\mathbf{e}$: uncorrelated zero-mean Gaussian noise

Denoised signal: $\tilde{\mathbf{x}} = \operatorname{argmin}_{\mathbf{x}} \left(\|\mathbf{x} - \mathbf{y}\|_2^2 + \alpha \, \mathrm{TV}(\mathbf{x}) \right) = \operatorname{argmin}_{\mathbf{x}} \left(\|\mathbf{x} - \mathbf{y}\|_2^2 + \alpha \|M\mathbf{x}\|_2^2 \right)$

We compare against the unweighted Möbius models.

Graphs

- Erdős–Rényi graphs
- Barabási-Albert graphs
- Thames river network

Signals

- Linear SEM signals (for Erdős–Rényi and Barabási-Albert graphs): $\mathbf{x} = A\mathbf{x} + \mathbf{n}$, $\mathbf{n} \sim \mathcal{N}(0, I)$
- Mean daily water flow in m^3/s

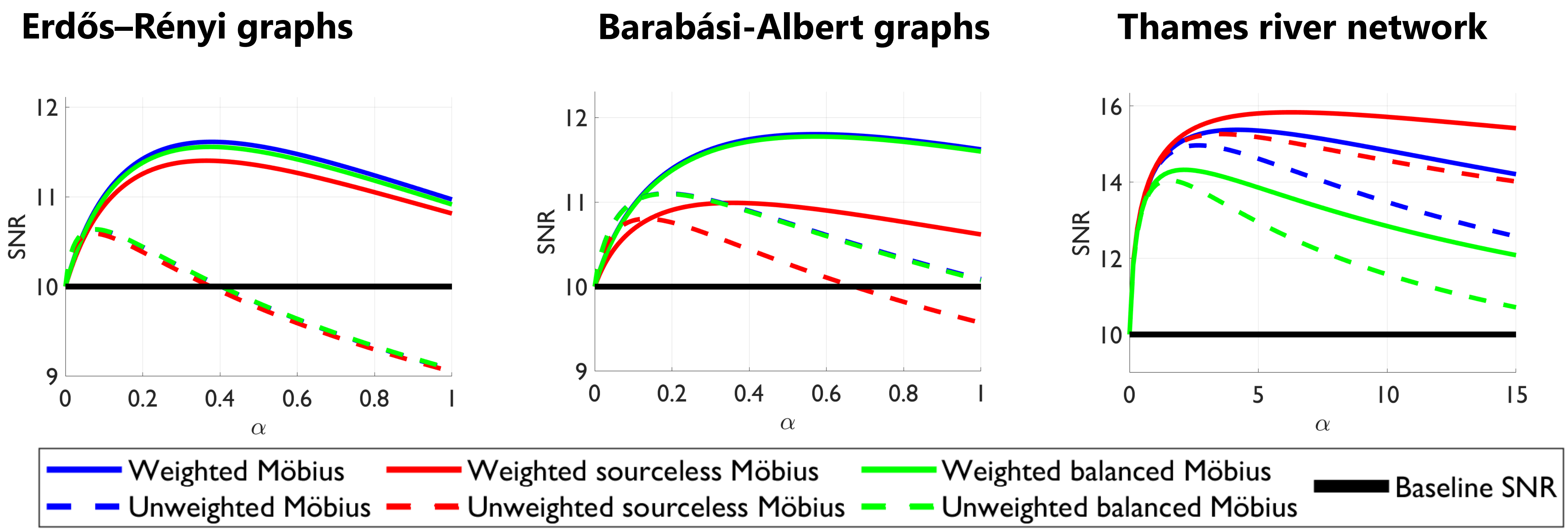
Edge weights

$\sim \mathcal{U}(0,1)$

$1, \quad 1 + \frac{\mu}{\text{distance}}$

Results

Erdős–Rényi graphs · Barabási-Albert graphs · Thames river network



Weighted Möbius models outperform the **unweighted** Möbius models.

Summary

- Novel orthogonal Fourier transform for weighted DAGs
- Generalizes the prior Möbius model for unweighted DAGs
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